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IDENTIFIERS	Advanced Mathematics; *Mathematical Thinking; Psychology of Mathematics Education; Representations (Mathematics); Situated Learning

ABSTRACT

This document, the first of three volumes, reports on the 15th annual conference of the International Group for the Psychology of Mathematics Education (PME) held in Italy 1991. Plenary addresses and speakers are: "Social Interaction and Mathematical Knowledge" (B. M. Bartolini); "Meaning: Image Schemata and Protocols" (W. Dorfler); "On the Status of Visual Reasoning in Mathematics and Mathematics Education" (T. Dreyfus); "The Activity Theory of Learning and Mathematics Education in the USSR" (T. Gabay). Research reports in this volume include: "Applications of R-Rules as Exhibited in Calculus Problem Solving" (Amit, M.; Movshovitz-Hadar, N.); "Effects of Diagrams on the Solution of Problems Concerning the Estimation of Differences" (Antonietti, A.; Angelini, C.); "Le probleme de statut du "millieu" dans un enseignement de la geometric avec support logiciel" (Artigue, M.; Belloc, J.; Kargiotakis, G.); "Procedural and Relational Aspects of Algebraic Thinking" (Arzarello, F.); "Hegemony in the Mathematics Curricula the Effect of Gender and Social Class on the Organisation of Mathematics Teaching for Year 9 Students" (Atweh, B.; Cooper, T.); "Instantaneous Speed: Concept Images at College Students' Level and its Evolution in a Learning Experience" (Azcarate, C.); "Students' Mental Prototypes for Functions and Graphs" (Bakar, M.; Tall, D.); "Illustrations de problemes mathematiques complexes mettant en jeu un changement au une sequence de changements par des enfants du primaire (Bednarz, N.; Janvier, B.); "The Operator Construct of Rational Number: A Refinement of the Concept" (Behr, M.; Harel, G.; Post, T.; Lesh, R.); "Children's Use of Outside-School Knowledge to Solve Mathematics Problems In-School" (Bishop, A. J.; De Abreu, G.); "Influences of an Ethnomathematical Approach on Teacher Attitudes to Mathematics Education" (Bishop, A. J.; Pompeu, G., Jr.); "Gender and the Versatile Learning of Trigonometry Using Computer Software" (Blackett, N.; Tall, D.); "La dimension du travail psychique dans la formation continue des enseignant(e)s de mathematiques" (Blanchard-Laville, C.); "Drawing a Diagram:

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Observing a Partially-Developed Heuristic Process in College Students" (Bodner, B. L.; Goldin, G. A.); "The Active Comparison of Strategies in Problem-Solving: an Exploratory Study" (Bondesan, M. G.; Ferrari, P. L.); "Teachers' Conceptions of Students' Mathematical Errors and Conceived Treatment of Them" (Boufi, A.; Kafoussi, S.); "Children's Understanding of Fractions as Expressions of Relative Magnitude" (Carraher, D. W.; Dias, Schliemann A.); "Dificultad en problemas de estructura multiplicative de comparacion" (Castro, Martinez E.; Rico, Romero L.; Batanero, Bernabeu C.); "Construction and Interpretation of Algebraic Models" (Chiappini G., Lemut E.); "Analysis of the Behaviour of Mathematics Teachers in Problem Solving Situations with the Computer" (Chiappini, G.; Lemut E., Parenti, L.); "Analysis of the Accompanying Discourse of Mathematics Teachers in the Classroom" (Chiocca, C.; Josse, E.; Robert, A.); "Van Hiele Levels of Learning Geometry" (Clements, D. H.; Battista, M. T.); "Some Thoughts about Individual Learning Group Development, and Social Interaction" (Cobb, P.); "Une analyse des brouillons de calcul d'eleves confrontes a des items de divisions ecrites" (Conne, F.; Brun, J.); "Brian's Number Line Representation of Fractions" (Davis, R. B.; Alston, A.; Maher, C.); and "Pupils' Needs for Conviction and Explanation within the Context of Geometry" (De Villiers, M.). This volume contains the addresses of research report authors. (MKR)

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INTERNATIONAL GROUP
FOR THE PSYCHOLOGY
OF MATHEMATICS EDUCATION

PROCEEDINGS

FIFTEENTH
PME CONFERENCE

ASSISI (ITALY) 1991
JUNE 29 - JULY 4

VOLUME I

Leonardo da Vinci, Quaderni di anatomia
(in: L. Villani, Leonardo a Milano, Musumeci Ed.)

INTERNATIONAL GROUP
FOR THE PSYCHOLOGY
OF MATHEMATICS EDUCATION

PROCEEDINGS

FIFTEENTH PME CONFERENCE

ASSISI (ITALY) 1991
JUNE 29 - JULY 4

VOLUME I

Published by the Program Committee of the 15th PME Conference, Italy.

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Editor: FULVIA FURINGHETTI
Dipartimento di Matematica dell'Università di Genova, Italy

The present volume is an anastatic copy in A4 format of the original edition of 1991

PROCEEDINGS OF PME 15, ASSISI (ITALY), 1991

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PME-XV PROCEEDINGS

edited by **Fulvia Furinghetti**
Italy

PREFACE

The first meeting of PME took place in Karlsrue in 1976. Thereafter different countries (Netherlands, Germany, U.K., U.S.A., France, Belgium, Israel, Australia, Canada, Hungary, Mexico) hosted the conference. In 1991 Italy will play host to the PME. The conference will take place in Assisi, a beautiful town of fine architectural and pictorical works of the Italian Middle Ages.

Italy has had a fine tradition of mathematics education since the second half of nineteenth century . In recent years the Italian mathematical community has shown an increasing interest in the themes which characterize the PME, and since 1988 there has been a significant number of Italian research reports .

Indeed there seems to be a growing worldwide interest in the conference and its themes,with papers from 24 countries included in these Proceedings and a wider range of delegates to the upcoming conference.

The academic programme of PME includes:

- 126 research reports
- four plenary addresses
- a panel
- 10 working groups
- 8 discussion groups
- 43 poster presentations.

The most relevant innovation with respect to the structure of the previous PME Conferences is the presence of a panel specifically focused on information technology in maths education. This fact constitutes a definitive acknowledgement of the importance of considering this topic as an integrated part of the elements to be taken into account in discussing the aspects of teaching and learning mathematics and the implications thereof.

A note on the research reports classification and emerging trends

The research reports are presented in these Proceedings in alphabetic order, with a further classification according to the following themes suggested by the International Programme Commitee. A maximum of three keywords was assigned to each paper.

1. Geometry and spatial thinking
2. Computers and mathematics learning
3. Algebraic thinking
4. Functions
5. Advanced mathematical thinking
6. Fractions, decimals, rational numbers, proportional reasoning
7. Imagery and visualization
8. Early mathematics learning
9. Proof
10. Problem solving
11. Pupils' conceptions, beliefs,...
12. Teachers' conceptions, beliefs,...
13. Social and affective factors, metacognition
14. Social construction of mathematics knowledge and linguistics
15. Out-of-school mathematics, the role of context,...
16. Assessment, evaluation
17. Theoretical and epistemological issues
18. Curriculum materials
19. Teachers education

This classification and the related attribution of a paper to a given area of interest has to be considered subjective and, consequently, as it happens to all classifications, may be discussed and modified. Nevertheless, it seems useful to organize the reported research activities and to help participants when selecting sessions to attend. Of course, the reading of the abstracts offers most personal information at this regard.

Through this classification, we have an approximate idea of the current trends of the research in the domain of PME. The data we collected confirm some aspects still observed in the previous edition; in particular, we see that the themes with the higher percentages are (in decreasing order) *problem solving; computers and mathematics learning; pupils' conceptions...; early mathematics learning; social construction of mathematics knowledge and linguistics; geometry and spatial thinking; theoretical and epistemological issues; teachers' conceptions....; teachers' education; algebraic thinking; fractions, decimals, rational numbers and proportinal resoning; imagery and visualization; social and affective factors,metacognition.*

The observations about the trends of current research are confirmed by the themes of the working groups and of the discussion groups.

More than in the past the general interest of the different activities scheduled in the conference seems focused not only on students' behaviour, but on the figure of the teacher and on the context and social factors intervening in the teaching learning processes.

A note on the review process

The Programme Committee received a total of 181 research proposals that encompassed a wide variety of theoretical and empirical approaches. Clearly - as already observed for preceeding meetings- the process of reviewing such a diverse collection of papers cannot be reduced to an algorithmic procedure. It is a process that copes with novelty and diversity by relying on situated wisdom and judgement. Nonetheless, a few general remarks can be made.

Each proposal was sent to three colleagues for review with the request that comments be provided and that these would be forwarded to the author(s).The review categories were:

A: definitively accept

B: accept with reservations

C: accept as poster

D: reject

The comments were forwarded to the authors along with the Programme Committee's decision.

List of the PME-XV reviewers:

Adda Josette, France
Artigue Michele, France
Balacheff Nicolas , France
Baroody Arthur, USA
Barra Mario , Italia
Behr Merlyn, USA
Bell Alan, U.K
Bergeron Jacques, Canada
Bishop Alan, UK
Booker George, Australia
Carraher David, Brazil
Cobb Paul, USA
Cooper Martin, Australia
Davenport Linda, USA
Doerfler Willibald, Austria
Douady Regine, France
Dreyfus Tommy, Israel
Eisenberg Ted, Israel
Davis Bob, USA
Fasano Margherita, Italia
Fernandez Domingo, Portugal

Ferrari Mario, Italia;
Filloy Yague Eugenio, Mexico
Furinghetti Fulvia, Italia
Gaulin Claude, Canada
Goldin Gerald, USA
Gutierrez Angel, Spain
Hanna Gila, Canada
Hart Kath, UK
Hart Lynn, USA
Herscovics Nicolas, Canada
Hiebert James, USA
Hitt Fernando, Mexico
Hoyles Celia, U.K.
Jaworski Barbara, UK
Jensen Ulla, Denmark
Kaplan Rochelle, USA
Kaput James, USA
Kieran Carolyn, Canada
Krummeheur Gotz, Germany
Laborde Colette, France
Leder Gilah, Australia
Leitao Rodriguez Ana Maria, Portugal
Lesh Dick, USA
Lester Frank , USA
Lin Fou Lai, Taiwan
Maher Carolyn, USA
Malara Nicolina, Italia
Matos Joao, Portugal
Matos José, Portugal
Meissner Hartwig, Germany
Mendicuti Teresa, Mexico
Nohda Nobuhiko, Japan
Noss Richard, UK
Nunes Terezinha, Brazil
Pimm David, UK
Pirie Susan, UK
Ponte Joao Pedro, Portugal

Puig Luis, Spain
Rojano Teresa, Mexico
Schliemann Analucia Diaz, Brazil
Simon Martin , USA
Speranza Francesco, Italy
Steffe Leslie, USA
Streefland Leen, Netherlands
Tirosh Dina, Israel
Vergnaud Gerard, France
Verhage Helen, Netherlands
Vinner Shlomo, Israel
Wood Terry, USA

INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

THE PRESENT OFFICERS OF THE GROUP ARE:

- President	Kath Hart (U.K.)
- Vice-Presidents	Gilah Leder (Australia), Fou Lai Lin (Taiwan)
- Secretary	Martin Cooper (Australia)
- Treasurer	Frank Lester (U.S.A.)

OTHER MEMBERS OF THE INTERNATIONAL COMMITTE ARE:

- Michele Artigue (France)
- Paul Cobb (U.S.A.)
- Theodore Eisenberg (Israel)
- Claude Gaulin (Canada)
- Angel Gutierrez (Spain)
- Gila Hanna (Canada)
- Barbara Jaworski (U.K.)
- Colette Laborde (France)
- Nobuhiko Nohda (Japan)
- David Pimm (U.K.)
- Joao Ponte (Portugal)
- Dina Tirosh (Israel)

PME-XV PROGRAMME COMMITTEE:

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- Paolo Boero (Italy)
- Elisa Gallo (Italy)
- Gila Hanna (Canada)
- Colette Laborde (France)
- Teresa N. de Mendicuti (Mexico)
- Michele Pellerey (Italy)
- David Pimm (U.K.)
- Joao Ponte (Portugal)
- Dina Tirosh (Israel)
- Vinicio Villani (Italy)

PME-XV LOCAL ORGANIZING COMMITTEE:

- Paolo Boero (Genoa)
- Francesca Conti (Perugia)
- Fulvia Furinghetti (Genoa)

HISTORY AND AIMS OF THE P.M.E. GROUP

At the Third International Congress on Mathematical Education (ICME 3, Karlsruhe, 1976) Professor E. Fishbein of the Tel Aviv University ,Israel, instituted a studying group bringing together people working in the area of the psychology of mathematics education. PME is affiliated with the International Commission for Mathematical Instruction (ICMI). Its past presidents have been Prof. Efraim Fishbein , Prof. Richard R. Skemp of the University of Warwick, Dr. Gerard Vergnaud of the Centre National de la Recherche Scientifique (C.N.R.S.) in Paris, Prof. Kevin F. Collis of the University of Tasmania , Prof. Pearla Nesher of the University of Haifa, Dr. Nicolas Balacheff, C.N.R.S. - Lyon.

The major goals of the Group are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
- To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Membership

Membership is open to people involved in active research consistent with the Group's aims, or professionally interested in the results of such research.

Membership is open on an annual basis and depends on payment of the subscription for the current year (January to December).

The subscription can be paid together with the conference fee.

RESEARCH REPORT CLASSIFICATION

1. Geometry and spatial thinking
M.Artigue, J. Belloc, G.Kargiotakis;
D.A.Clements, M.T.Battista;
M. De Villiers;
L.D. Edwards;
A. Gutierrez, A. Jaime, J.M. Shaughnessy, W.F. Burger;
K. Krainer;
C. Kynigos;
F. Lopez- Real ;
S. Magina, C.Hoyles;
M.A.Mariotti ;
A.L.Mesquita;
R. Nunez Errazuriz;
I.Osta;
F.Pluvinage, J.C. Rauscher, C. Dupuis;
R. Strasser, B. Capponi;
G.H. Wheatley, A.Reynolds;
O.Zaslavsky .

2. Computers and mathematical learning
M.Artigue, J. Belloc, G.Kargiotakis;
N. Blackett, D. Tall ;
G.Chiappini, E.Lemut, L.Parenti;
L.D. Edwards;
N.Hativa, O.Pomeranz, S. Hershkovitz, I. Mechmandarov;
L. Healy, S. Pozzi, C.Hoyles;
R. Hershkovitz, A. Friedlander, T. Dreyfus;
C.Hoyles, R.Sutherland, R. Noss;
J.J. Kaput, C. Hancock;
G. Krummheuer;
C. Kynigos;
T. Lemerise, B. Coté;
R. Lesh, A.E.Kelly;
N.A.Malara,R.Garuti;
R. Nemirovsky, A. Rubin;

L. Nevile;
R. Noss, C.Hoyles;
I.Osta;
M.Reggiani;
T.Rojano, R.Sutherland;
B.Schwarz, T.Dreyfus;
I. Stevenson, R. Noss;
R. Strasser, B. Capponi;
M. Yerushalmy, R. Gafni;
E. Wenzelburger Guttenberger;

3. Algebraic thinking

F. Arzarello;
G. Chiappini, E. Lemut;
E.Filloy-Yague ;
N. Herscovics, L.Linchevski;
C.Kieran;
L.Linchevski, A. Sfard;
C.Margolinas;
T.C. O'Brien;
I.Peled;
T.Rojano, R.Sutherland;
D.Tirosh, R. Hadass, N. Movshovitz-Hadar;
S. Ursini;
M.A. Wolters;

4. Functions

Md Nor Bakar, D. Tall;
R. Even, Z. Markovitz;
G.A.Goldin, N. Herscovics;
G. Harel, E. Dubinsky;
A. van Streun;
E. Wenzelburger Guttenberger;

5. Advanced mathematical thinking

M.Amit, N. Movshovitz-Hadar;
C.Azcarate;

L.D. Edwards;
G.A.Goldin, N. Herscovics;
J.J. Kaput, C. Hancock;
F.L.Lin;
R. Nemirovsky, A. Rubin;
G. Shama, T. Dreyfus;

6. Fractions, decimals, rational numbers, proportional reasoning

M.Behr, G.Harel, T.Post, R.Lesh;
D.W. Carraher, A.Dias Schliemann;
R.B. Davis, A. Alston, C. Maher;
L.Grugnetti, C. Mureddu Torres;
G. Harel, M.Behr, T.Post, R. Lesh;
C.Hoyles, R.Sutherland, R. Noss;
T. Lemerise, B. Coté;
F.L.Lin;
S. Llinares Ciscar, M. V. Sanchez Garcia;
D.Neuman;
S. Ohlsson, N.Bee;
A.Saenz-Ludlow;

7. Imagery and visualization

A. Antonietti,C.Angelini ;
Md Nor Bakar, D. Tall;
B.L. Bodner,G.A. Goldin;
R. Hershkovitz, A. Friedlander, T. Dreyfus;
F.Lopez-Real
M.A. Mariotti;
T.C. O'Brien;
N.C. Presmeg;
G. Shama, T. Dreyfus;
M. Yerushalmy, R. Gafni;

8. Early mathematical learning

D.W. Carraher, A.Dias Schliemann;
E.Castro Martinez, L.Rico Romero, C.Batanero Bernabeu, E. Castro Martinez;
F. Conne, J. Brun;

K. Hasemann;
N. Herscovics, L.Linchevski;
K. Ito;
G.C. Leder;
C.A.Maher, A.M. Martino;
T. Nunes, P. Light, J. Mason;
L.Morgado;
H.Murray, A. Olivier, P. Human;
D. Neuman;
T.C. O'Brien;
M. Orozco Hormaza;
I.Peled;
L.Poirier, N.Bednarz;
A.Saenz-Ludlow;
G.B. Saxe;
G.H. Wheatley, A.Reynolds;
M. A. Wolters;

9. Proof

M. De Villiers;
F. Furinghetti,D.Paola;
M.Miyazaki;
N.Movshovitz-Hadar;

10. Problem solving

A.J.Bishop,G.de Abreu ;
M.Amit, N. Movshovitz-Hadar;
A. Antonietti,C.Angelini;
N.Bednarz, B. Janvier;
B.L. Bodner,G.A. Goldin;
M.G.Bondesan, P.L.Ferrari;
E.Castro Martinez, L.Rico Romero, C.Batanero Bernabeu, E. Castro Martinez;
B.Greer, M. McCann;
U. Grevsmuhl;
L.Grugnetti, C. Mureddu Torres;
J.L. Gurtner,B.Vitale;
N.Hativa, O.Pomeranz, S. Hershkovitz, I. Mechmandarov;

G.C. Leder;
P. Nesher, S. Herskovitz; G. Krummheuer;
N.A.Malara,R.Garuti;
L.Morgado;
H.Murray, A. Olivier, P. Human;
P.Nesher, S. Hershkovitz;
M. Orozco Hormaza;
L.Poirier, N.Bednarz;
M.Reggiani;
T.Rojano, R.Sutherland;
K. Schultz;
K.Stacey;
R. Stavy, D. Tirosh;
A. van Streun;
T. Wood;

11. Pupils' conceptions, beliefs,...

D.W. Carraher, A.Dias Schliemann;
F. Conne, J. Brun;
R.B. Davis, A. Alston, C. Maher;
M. De Villiers;
F. Furinghetti, D. Paola;
B.Greer,M.Mc Cann;
L.Grugnetti, C. Mureddu Torres;
J.L. Gurtner,B.Vitale;
K. Hasemann;
K. Ito;
K. Krainer;
E.Love, C. Shiu;
S. Magina, C.Hoyles;
C.A.Maher, A.M. Martino;
T. Nunes, P. Light, J. Mason;
R. Nemirovsky, A. Rubin;
D. Neuman;
M.J. Perrin-Glorian;
T.L.Schroeder;
K.Stacey;

O. Zaslavsky

12. Teachers' conceptions, beliefs,...

A.J.Bishop, G.Pompeu;
A.Boufi, S.Kafoussi;
G.Chiappini, E.Lemut, L.Parenti;
R. Even, Z. Markovitz;
G. Harel, E. Dubinsky;
B.Jaworski;
M. Jurdak;
R. Lesh, A.E.Kelly;
S. Llinares Ciscar, M. V. Sanchez Garcia;
C.Moreira;
J.A. Mousley;
F.Pluvinage, J.C. Rauscher, C. Dupuis;
N.C.Presmeg;
T.L.Schroeder;
M.A. Simon;
D.Tirosh, R. Hadass, N. Movshovitz-Hadar;

13. Social and affective factors, metacognition

B. Atweh, T. Cooper ;
N.Blackett,D.Tall;
T. Eisenberg;
J.T.Evans;
U. Grevsmuhl;
L.C. Hart;
C. Kanes;
S. Lerman, R. Scott-Hodgetts;
M.J. Perrin-Glorian;
D.Pimm;

14. Social construction of mathematical knowledge and linguistics

M.G.Bondesan, P.L.Ferrari;
P.Cobb;
N.F. Ellerton;
U. Grevsmuhl;

C. Chiocca,E. Josse, A. Robert;
L. Healy, S. Pozzi, C.Hoyles;
C. Kanes;
G. Krummheuer;
L. Lindeskov;
F. Lopez-Real;
C.A.Maher, A.M. Martino;
C.F. Newman ,
D.Pimm;
S. Pirie;
S.Pirie, T. Kieren;
H. Sakonidis,J.Bliss;
E. Yackel;
T. Wood;

15. Out-of-school mathematics, the role of context,...

A.J.Bishop;
A.J.Bishop, G.Pompeu;
G. de Abreu;
J.T. Evans;
L.Lindeskov;
G.B. Saxe;
A. van Streun;

16. Assessment, evaluation

D.A.Clements, M.T.Battista;
A. Gutierrez, A. Jaime, J.M. Shaughnessy, W.F. Burger;
E. Love, C.Shiu;
J.A. Mousley ;
F.Pluvinage, J.C. Rauscher, C. Dupuis;
B.Schwarz, T.Dreyfus;

17. Theoretical and epistemological issues

M.Artigue, J. Belloc, G.Kargiotakis;
R.B. Davis, A. Alston, C. Maher;
P. Ernest;
G.A.Goldin, N. Herscovics;

E. Gray, D. Tall;
J.L. Gurtner, B. Vitale;
N. Hall;
B. Jaworski;
M. Jurdak;
L. Linchevski, A. Sfard;
L. Lindeskov;
N. Movshovitz-Hadar;
C. Margolinas;
C.F. Newman, S. Pirie;
R. Nunez Errazuriz;
S. Pirie, T. Kieren;
K. Schultz;

18. Curriculum materials

K. Stacey;

19. Teachers education

C. Blanchard-Laville;
A. Boufi, S. Kafoussi;
G. Chiappini, E. Lemut, L. Parenti;
C. Chiocca, E. Josse, A. Robert;
R. Even, Z. Markovitz;
G. Harel, E. Dubinsky;
L.C. Hart;
S. Lerman, R. Scott-Hodgetts;
R. Lesh, A.E. Kelly;
S. Llinares Ciscar, M. V. Sanchez Garcia;
L. Nevile;
R. Noss, C. Hoyles;
M.A. Simon;
D. Tirosh, R. Hadass, N. Movshovitz-Hadar;
M. Yerushalmy, R. Gafni;

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Working Group on "Social Psychology of Mathematics Education"

All mathematics learning takes place in a social setting and particularly within the PME community, we need to be able to interpret, and theorise about, mathematics learning interpersonally as well as intrapersonally. Mathematics learning in its educational context cannot be fully interpreted as an intrapersonal phenomenon because of the social context in which it occurs. Equally, interpersonal or sociological constructs will be inadequate alone since it is always the individual learner who must make sense and meaning in the mathematics. Therefore, it is vitally important to research the ways this intra-interpersonal complementarity influences the kind of mathematical knowledge acquired by pupils in classrooms. In order to pursue this research it is therefore necessary to analyse and develop both theoretical constructs and methodological tools.

This is what the SPME working group has concentrated on. At PME 10, the first official meeting of the group, we tried out various small group tasks amongst ourselves and discussed their value as research 'sites' and also teaching situations. At PME 11, we moved to other social determinants of mathematics learning, particularly thinking about influences of other pupils and of the teacher. At PME 12, we focussed on the idea of "bringing society into the classroom" and the issues of justifying research which might conflict with what "society", considers education should be doing. At PME 13, we worked on two areas, firstly the ways in which the construct "mathematics" is socially mediated in the classroom, and secondly, the use of videos of classroom interactions, and their analyses. At PME 14, we considered the situation of bi-cultural learners, the social setting of the nursery-school, and the learning values of cooperative games.

At PME 15, we will consider the following:

(1) bi-cultural learners - particularly ideas from the evidence of Guida de Abreu from Brazil;

(2) the relationships between the social contexts of mathematics and the child development model, led by Leo Rogers from UK;

(3) social issues of assessment, let by Luciana Bazzini and Lucia Grugnetti from Italy, and

(4) aspects of cultural and social 'difference' which may be of significance in mathematics learning.

Alan J Bishop

PME XV Working Group

Psychology of Inservice-education of Mathematics Teachers - a Research Perspective

Sandy Dawson - Simon Fraser University, Vancouver, Canada
Barbara Jaworski - University of Birmingham, UK
Terry Wood - Purdue University, Indiana, USA

This group is concerned with the development of mathematics teaching. Its work so far may be summarised as follows:

1. Discussion of particular inservice projects with mathematics teachers in which PME participants are, or have been, engaged, leading to identification of questions and issues both of inservice itself, or of research relating to it.

2. Attempting to refine questions and issues raised in (1) with regard to emergent theoretical frameworks. Consolidating group understanding and rationalising individual perspectives.

3. Looking to the future. What work may be undertaken at and between PME meetings? Members declare their likely involvement with issues raised and possibly agree to write papers for circulation. Possibilities include consideration of research methodologies and potential theoretical frameworks.

Issues raised have included:

1 **Expectations** - possible mismatches of expectations between leaders and participants in INSET events or courses. How can the degree of compatibility of expectations of course participants with the aims of the course affect what occurs? For example, how might a provider's position, perhaps in terms of *not giving 'answers', encouraging self development and the 'active' learner*, be married to some participants' positions, perhaps of *wanting answers, "tell me what to do", the passive learner, the provider as 'expert'*? How do participants perceptions of mathematics and its teaching influence expectations?

2 **The teacher-educator's dilemma** - flexibility versus control. How open is it possible to leave activities, while still hoping to achieve objectives. If objectives are directly sought, can they be achieved? Is it possibe to share objectives with participants? Can participants understand a leader's objectives before having gained the awareness or experience which the course sets out to engender?

3 **Implementation** - the danger that 'nothing happens' after return to normal work. How might participants be supported in the implementing of ideas and awareness developed as part of the course? Which comes first - implementation or philosophical change? Do teachers need to start to modify their philosophical base before they can make changes to their practice? Or can starting to make changes trigger awareness, and thus promote a change of philosophy? Or do the two things occur simultaneously? How might answers to these questions affect the way courses are structured?

4. What **methodologies - processes, structures, techniques** of INSET do we employ?

(For more information please contact Barbara Jaworski: School of Education, University of Birmingham, Birmingham B15 2TT, UK)

RESEARCH ON THE PSYCHOLOGY OF MATHEMATICS TEACHER DEVELOPMENT

he Working Group *Research on the Psychology of Mathematics Teacher Development* was first nvened as a Discussion Group at PME X in London in 1986, and continued in this format until e Working Group was formed in 1990 at PME XIV in Mexico. This year, at PME XV, we have e opportunity to build on the foundation of shared understandings that have been established.

ims of the Working Group

The development, communication and examination of paradigms and frameworks for research in e psychology of mathematics teacher development

The collection, development, discussion and critiquing of tools and methodologies for nducting naturalistic and intervention research studies on the development of mathematics achers' knowledge, beliefs, actions and thinking

The implementation of collaborative research projects

The fostering and development of communication between participants

The production of a joint publication on research frameworks and methodological issues within is research domain.

esearch Questions

he following are examples of the types of research questions that will be addressed at PME XV:

What are the implications of ethnomathematics/everyday cognition research for preservice teacher education and for professional development programs for teachers?

To what extent do mathematics educators apply constructivist tenets in their everyday practice of teacher education? Is there a mismatch between their ideals and their practice?

What should be done to help young pre-service students who lack confidence in their mathematical ability and in their ability to teach mathematics?

Why is it that most young mathematics teachers become progressively more conservative as they gain classroom experience?

roposed Outcomes of the Working Group at PME XV

Collaborative Research Projects: Members of the Working Group have overlapping esearch interests, and it is hoped that collaborative research projects can be mounted. This would ave the benefit not only of enriching the data available, particularly where such collaborative esearch could be carried out in different countries, but also of accessing a wider experience base or the development of suitable instruments and for subsequent analysis.

Publication of an Edited Collection of Research Papers: It is proposed that the Vorking Group publish a collection of research papers that address a single theme (one of the bove research questions, for example).

Nerida Ellerton, Convenor

WORKING GROUP on ADVANCED MATHEMATICAL THINKING (A.M.T.)

Organisers: Gontran Ervynck, David Tall

SESSION I: THE RELATIONSHIP BETWEEN PROCESS, CONCEPT AND DEFINITION. Initiator: David Tall (U.K.).

We have noted the way in which some students cling to processes (e.g. counting in arithmetic) which do not become encapsulated as concepts which can then be manipulated at a higher level. We have found that the *symbolism* is very important because throughout mathematics there are instances of the same notation being used for process and output of the process: e.g. $^3/_4$; $f(x)$; $\lim(a_n)$; etc. The mathematician uses the same symbolism to represent both process and output concept. It is this ambiguity and flexibility that makes mathematical thinking so powerful. We have coined the term "procept" to be the combination of process and concept which are represented by the same symbolism. We are at present working on the consequences at all levels of mathematical development, including AMT.

SESSION II: THE TRAINING OF RESEARCHERS IN MATHEMATICS EDUCATION. Initiators: Gontran Ervynck (Belgium), Ted Eisenberg (Israel).

We are concerned about the training of *researchers in mathematics education* versus doctoral programs. What should we require, not only in the way of *research* but also in the way of *course work*? The world of mathematics is changing, but many recent graduates don't seem to be aware of these changes. A shift in the understanding of the concepts (creation of new, appropriate concept images) is needed but often inexistent. They are not researchers in mathematics, but they should develop a suitable psychological approach to the subject. We think it would be nice to at least discuss the issue. Ultimately, we would not mind seeing a recommended curriculum as the result of a discussion in the AMT Group.

SESSION III: THE PROCESS BY WHICH STUDENTS COME TO APPRECIATE THE NEED FOR RIGOR AND FORMALITY. Proposed by Steve Williams (U.S.A.).

This differs from "proof" in that concepts and procedures may also be addressed with formality and rigor (e.g. the formal definition of limit). This topic may also involve a discussion of students' views of what constitutes mathematics, their epistemological committments, their beliefs about generality. We see this as an important complement to current efforts to make mathematics intuitive and contextual; while "practical" mathematics may be didactically important, it is also important to study how the movement toward rigor is made and mathematical maturity is gained.

SESSION IV: THE WORKING OF THE GROUP IN THE FUTURE.

In the last session we will discuss as usual plans for the future work of the Group. In particular suggestions and volunteers for an AMT-program for next year are welcome.

The IGPME Working Group on Representations

Gerald A. Goldin, Center for Mathematics, Science, and Computer Education,
Rutgers University, New Brunswick, New Jersey 08903, USA

The Working Group on Representations of the International Group for the Psychology of Mathematics Education was guided during its first several years by Frances Lowenthal (Mons, Belgium). I took on the responsibility of coordinating it at the end of the 1989 meeting in Paris, and Claude Janvier (Montreal, Canada) provided valuable assistance in organizing our sessions at the 1990 meeting in Mexico, where we had 42 participants. In the last year our focus has been broadened substantially, to include a number of different interpretations that have been given to the term "representation" in connection with mathematics learning, teaching, and development:

(a) *External physical embodiments (including computer environments):* An external, structured physical situation or set of situations that can be described mathematically or seen as embodying a mathematical idea--e.g., a number line, drawn and labeled; a configuration of pegs on a peg-board providing an array model for multiplication, or more broadly the peg-board apparatus itself; a calculator- or computer-environment where constructs such as functions or graphs can be displayed and manipulated; etc. (b) *Linguistic embodiments:* Verbal, syntactic, and related semantic aspects of the language in which problems are posed and mathematics is discussed. (c) *Formal mathematical constructs:* A different meaning of "representation", still with emphasis on a problem environment external to the individual, is that of a formal structural or mathematical analysis of a situation or set of situations--e.g., state-space representations of problems or games such as the Tower of Hanoi, Nim, etc.; representations of mathematical entities, such as groups, rings, functions, etc., by means of other mathematical entities, such as operators on vector spaces, graphs, etc. Although there is a sense in which all mathematics can be regarded as "internal," the emphasis here is on "representation" as an analytical tool for formalizing or making precise mathematical ideas or mathematical behavior. (d) *Internal cognitive representations:* Very important emphases include students' internal, individual representation(s) for mathematical ideas such as "area", "functions", etc., as well as systems of cognitive representation in a broader sense that can describe the processes of human learning and problem solving in mathematics. These and other meanings of the term "representation" were addressed in at least a preliminary way in the book, *Problems of Representation in the Teaching and Learning of Mathematics* (Erlbaum, 1987), ed. by Claude Janvier; see also the articles therein by James Kaput.

The scope of the Working Group on Representations includes many related questions. The following are typical: What are the consequences of creating and manipulating particular external representations of mathematical concepts? How can we develop new external systems of representation that foster more effective learning and problem solving? How can we describe in detail the internal cognitive representations of learners and problem solvers? What is the nature of the interaction between external and internal representations? How do we *infer* internal representations by observing external behaviors? How do individuals construct internal representations from their experience of external environments? What can *theories* based on cognitive representation tell us about making mathematics education more effective? What is the role of metaphor in cognitive representation in mathematics? Such questions are addressed not only as general, abstract considerations, but in the particular contexts of mathematical activity in specified domains, and with various populations of students, teachers, mathematicians, and so forth. A variety of perspectives are reflected in the Working Group; we are presently aiming toward a special volume of the *Journal of Mathematical Behavior* that will be devoted to the topic of "representation".

Teachers as Researchers in Mathematics Education

Co-ordinators: Stephen Lerman & Rosalinde Scott-Hodgetts

The group has been meeting since PME12 in Veszprem, first as a discussion group and subsequently as a Working Group. The aims of the group are to review the issues surrounding the theme of teachers as researchers in mathematics education, and to engage in collaborative research.

The justification for the notion that *classroom teachers* can and should carry out research, whilst they are involved in the practice of teaching mathematics, comes from a number of sources, including: notions of teachers as reflective practitioners; teaching viewed as a continuous learning process; the nature of the theory/practice interface; the problems of dissemination of research when it is centred in colleges; the advantages of research problems being generated in the classroom and finding solutions within the context in which the questions arise. These themes are seen to be equally relevant to teacher education, and provide a focus for our own reflective activities as teacher-educators.

At this meeting in Assissi, the programme will be centred around reports, by members of the group, on some work we have been doing. Individual group members have been using similar methodologies, in which students on in-service courses have been encouraged to engage in action research in their classrooms. We will be reviewing the nature of the students' learning, through this kind of programme, and our own learning, in relation to our in-service work. We will also continue discussions on the themes mentioned above, and identify areas of study for the coming year.

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Ratio and Proportion

F.L.Lin, K.M.Hart and J.C.Bergeron

Is the research topic 'ratio and proportion' dead or alive?

To tackle this question, this group have tried to define what is proportional reasoning abilities and found that what we have known is incomplete.

In the Mexico meeting, the group have worked on some very fundamental questions, such as

(1) what is fractions? Is $\pi/2$ a fraction?
(2) what is ratio? Is $a:b = 3:4$ a ratio?
(3) what is the relation between ratio and fraction?

Some members of the group planned to study these questions during this year. The group will spend one time slot on these questions.

On the other end, the group have also worked on advanced proportional reasoning. Discussion on some recent development and problems for further investigation in this direction will take one time slot over.

The third time slot will work on questions, such as

(1) what is the origin of fraction/ratio concepts?
(2) how to develop diagnostic teaching module on beginning fractions/ratio? ... etc.

Welcome to ratio and proportion group.

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GEOMETRY WORKING GROUP

Helen Mansfield
Curtin University of Technology, Western Australia

The Geometry Working Group will have the overall theme **Learning and Teaching Geometry: a Constructivist Point of View** for PME XV in Italy. This theme was chosen because the committee of the Geometry Working Group believes that it is timely to examine constructivism as a theoretical framework for research into aspects of teaching and learning geometry.

Within this overall theme, two sub-themes will be discussed. These are **What Constructivism has to say about Learning and How Teaching can Promote Learning in Geometry**; and **Helping Students to Construct Knowledge in the Geometry Classroom.**

In the first session of the Geometry Working Group there will be introductory presentations concerning the first subtheme followed by a discussion. The focus in the second session will be on the role of computer environments in the learning and teaching of geometry.

The third session will provide opportunities for group members to present brief papers on their current research. These papers do not have to report work that is completed, but will provide opportunities for the presenters to discuss work in progress, to seek feedback from other participants, and to discuss with colleagues collaborative research projects.

Working Group on Algebraic Processes and Structure

Coordinator: Rosamund Sutherland

Institute of Education University of London

The group aims to characterise the multiple "jumps"/shifts that appear to be involved in developing an algebraic mode of thinking and to investigate the role of symbolising in this development. Other concerns of the group are the role of meaning in algebraic processing; the potential of computer-based environments and implications for classroom practice. Key issues will be discussed and worked on in small groups with the aim of producing a set of questions and working hypotheses for future collaboration.

New members are encouraged to join the group during PME XV.

CLASS ROOM RESEARCH

Jan van den Brink

History

The PME-workinggroup Class Room Research was founded as a discussiongroup at the 12th International Conference for the Psychology of Mathematics Education in Hungary in 1988.

Problems, standpoints and purposes of the group

A. Problems

1. In our research we have encountered similar methodological problems arising from our developmental approach to classroom research. One of the problems is how to analyse the classroom data collected; an important aspect of our research being the defining of new variables for each new set of data.

2. Our research has raised many question including: repeatability, generalizability, falsification, objectivity, qualitative/quantitative aspects, reduction of collected data, developing new research methods and techniqes, close to educational situations

B. Standpoints

The following perspectives are implicite in our research:

1. *Classroom Descriptors*

As well as describing the data collected from various activities presented to and/or undertaken by the children, we record key classroom descriptors. In particular, details of actual instruction given by the teachers is noted.This is not common in research where any instruction involved is mostly described in global terms.We have found, however, in all our work that the type of instruction given by the classroom teacher can be a distinguishing feature in the data collected from the children.

2. *Mood conditions*

One of the key issues to be considered by the group will involve searching for education conditions which produced a suitable mental climate for the children to work towards their *own* productions. They have to bring the children into the *mood* to do so. These conditions are mostly of a *social* character and help to legitimise the particular children's activities; the activities make *sense* to the children.

3. *Source*

We consider the educational setting in the classroom (the manner of teaching, and so on) to be a *source of techniques and methods for the researcher*.

4. *Mutual nature of the research*

Our focus is on *mutual* research situations in which the children can recognise themselves (e.g. as a writer, as an author, etc.). This is seen as important as it helps to justify the reseach objects *C.*

C. Purposes of the working group

To prepare a booklet to support researchers working in the area of classroom research and closely connected with the practice of teaching.

Language: English

Discussion groups

BAZZINI L.; GRUGNETTI L. - *Meaningful contexts for school mathematics* p. xxix

BELL A. - *The design of teaching* p. xxx

DENIS B.; LACASSE R.; PARZYSZ B. - *Learning mathematics and cultural context* p. xxxi

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HANNA G.; BALACHEFF N.; PIMM D. - *Theoretical and practical aspects of proof* p. xxxiii

HART K. - *The future scientific orientation of PME* p. xxxiv

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MEANINGFUL CONTEXTS FOR SCHOOL MATHEMATICS

Luciana BAZZINI,Dipartimento di Matematica, Università di Pavia (Italy)
Lucia GRUGNETTI, Dipartimento di Matematica, Università di Cagliari (Italy)

The word "context" can take different meanings in mathematical education: for instance it can stand in general for the socio-cultural or ethno-anthropological background or, more specifically, for the set of conditions in which teaching-learning processes take place.

In Chevallard's words, we are interested in "re-framing mathematical knowledge" by means of suitable contexts in the view of an expanding development of knowledge itself. For our purposes, *"context"* means *"the set of environmental conditions and experiences created or evoked in the classroom in order to give a meaning to the mathematical contents to be taught through problem situations concerning them"*

Our search for meaningful contexts for school mathematics has a double source: on the one hand, the growing relevance of this topic in the international debate (Lesh, 1985; Treffers & Goffree, 1985; Janvier, 1986; Carraher, 1988; Bishop, 1988; Boero, 1989); on the other hand, the specificity of the Italian teaching system which allows the systematic development of contextualized investigations for a long time (even years) , especially in primary and comprehensive schools.With regard to the Italian situation, the contingent fact that in Italy a teacher usually teaches the same class for 3 or 5 years reflects deeply-rooted cultural and educational choices aiming at a systematic development of the teaching-learning process.All this has led to looking for meaningful themes in which the student's mathematical activity can start and proceed according to a spiral and fan-shaped development.

There is a wide range of studies and experiences dealing with the contextualization of mathematical experience through various contexts (linked to the child's life,or related to other disciplines, such as economics, statistics, computer science and history).Experience has shown that suitable contexts are not only a "stimulus to learn", but they are also powerful suppliers of means apt to help students in creating strategies and in conceptualizing them.On the other hand, there is the risk that a strong context can thwart the concepts' transfer to other contexts and, eventually, limit the abstraction process.Investigating within a context needs a successive decontextualization and a recontextualization according to a cycle which continues in time (cfr.Douady,1985).

This Discussion Group is oriented mainly in docusing on the following questions:

- analysis and comparison of studies dealing with experiences carried out in school and related to the exploitation of suitable contexts for mathematical education
- analysis of the "two sides" of the coin "context", i.e. as a powerful opportunity for mathematical investigation or as a potential obstacle to abstraction
- discussion about possible implications for teaching at different age levels

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THE DESIGN OF TEACHING

Alan Bell, Shell Centre for Mathematical Education
University of Nottingham
England

Introduction

In view of the substantial amount of research which now exists on students' understanding of mathematics, and the somewhat smaller number of teaching experiments, it is appropriate to ask how far the results of this activity are now incorporated in the published teaching material which is the basis of most students' mathematical experience.

One might expect to see two main influences, first in the provision of tasks which focus attention on the key concepts, misconceptions and problem structures which research has shown to constitute the major learning points; and secondly that the learning activities might reflect up to date awareness of criteria for effective learning in mathematics in general.

The main aim of the discussion group is to review the set of pedagogical principles which might be regarded as now established by research and to apply them to some examples of recently published teaching material.

A Set of Principles for Designing Teaching

1. Richly connected bodies of knowledge are well retained; isolated elements are quickly lost.
2. Discussion of a few hard critical problems is more effective than progress through a sequence of many easy questions covering the same field: especially for retention.
3. Pure practice increases fluency but does not develop understanding.
4. Successful problem solvers monitor their progress both locally and globally.
5. Each performance of an action strengthens its memory trace. So it is important that, particularly in consolidation work, the learner has the means of knowing immediately whether his answer is right. Briefly, there needs to be immediate feedback either in the form of a check of the answer or through a discussion.
6. Scope for pupil choice and creative productions can provide both motivation and challenge at the pupils' own level.
7. Establishment of multiple connections is helped by exploring fully the relationships in one context before moving to another context.
8. Exploring the transfer of relations from one contact to another enhances their abstraction and their availability for further transfer.

Members are invited

1. to bring a few examples of teaching material with comments in the light of either these principles, or their own preferred set.
2. to consider this set of principles and suggest amendments;

LEARNING MATHEMATICS AND CULTURAL CONTEXT

Bernadette Denys, Equipe Didirem, Université de Paris 7.
Equipe de didactique des mathématiques, Université de Grenoble, France
Raynald Lacasse, Université d'Ottawa, Canada
Bernard Parzysz, Institut de Recherche pour l'Enseignement des Mathématiques,
Université de Paris 7, France

The process of learning mathematics takes place in the school environment; however this educational process cannot be isolated from the effects of the cultural context in which the child develops.

Intercultural research appears to show differences which reside in how certain cognitive processes operate in different socio-cultural contexts.

The study of the cultural context of learning mathematics is related to various disciplines and could require an interdisciplinary approach.

Some various researches and approaches of learning mathematics could be very relevant for arising questions about cognitive processes in relation with cultural aspects.

In the first two Discussion Group meetings in PME XIII and PME XIV, participants attended, from Australia, Belgium, Brazil, Canada, France, Germany, United Kingdom, Greece, Holland, Jamaica, Japan, Mexico, Switzerland and U.S.A.

We have focused the theme on the question

WHAT IS CULTURE IN LEARNING MATHEMATICS?

In relation with this question, we intend to present and discuss the following subjects:

- **The role of language** (using the problem of French and English in Canada);
- **Mastering space and learning geometry** (a Franco-Japanese Cooperative Research in hand).

Other questions could be introduced at this session, if submitted to the coorganizers.

Some of our interests are common with the *Working Group* organized by Alan Bishop et al. ***Social psychology of mathematics education***. A common session will be held with this Working Group, using some researches about the following question:

In our mathematical education studies in relation with cultural and / or social aspects, are these aspects taken as a starting point or an end-product?

We intend to become a *Working Group* in PME XVI.

Two time-slots will be allocated to the Discussion group
Monday morning 1st and Wednesday afternoon 3rd of July
One more time-slot will be common with the Working Group "Social psychology of mathematics education"

MODERN ART AND THE LEARNING OF MATHEMATICS

Ulrich GREVSMÜHL,
Pädagogische Hochschule Freiburg, Germany

The international movement of constructive art originated at the beginning of the 20th century in the De Stijl movement in Holland, in the German Bauhaus and in the East European constructivism. The constructivists have made it as their aim to investigate the principles of orders and structures in nature, in our society and in our technological environment and to express these in visual form.

The works of constructive art are of special interest for the teaching and learning of mathematics because of their mathematical, perceptional and conceptional implications. Today many investigations are undertaken in art with systems that are based on mathematical or scientific concepts. In general, the systems consists out of modules, onto which certain generative principles or functions act. The final work or a series of works is often the result of an intensive research work carried out by the artist.

In the discussion group several works of art will be presented in form of slides which may then serve as starting points for discussing various implications in the learning of mathematics. The following themes are proposed:

- Visual language and mathematical learning,
- Constructive artists, mathematicians and scientists: working approaches and modes of thinking,
- Perception, mental images, concept formation and concept representation,
- Perception, aesthetics and information processing.

References:
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THEORETICAL AND PRACTICAL ASPECTS OF PROOF

Gila Hanna
Canada

Nicolas Balacheff
France

David Pimm
England

In the last two decades several mathematicians and mathematics educators have questioned the tenant that mathematics consists mostly of formal manipulations and reasoning by formal deductions. In particular, the realization that in practice mathematicians admit that proofs can have different degrees of proofness, and further, that when a proof is valid by virtue of its form only, without regard to its content, that proof is bound to add very little to understanding of a particular topic and ironically might not necessarily be very convincing. Further, in mathematical practice it is often the significance of what is proved rather than the correctness of a proof, that is given more weight in the essentially social process of negotiation of meaning by which mathematical ideas gain acceptance. Most mathematicians choose to proceed with their work unconcerned with very formal proofs. It would therefore appear that what needs to be conveyed to students is the importance of careful reasoning and of building arguments that can be scrutinized and revised. This process, while involving formalization, places a great emphasis on content and ideas rather than on form only.

Some of the objectives of the working group will be: the examination of the role of proof, formal or not, in the mathematics curriculum; the discussion of research projects dealing with the concept of proof as presented in the classroom and understood by students; the identification of issues relating to the teaching of proof as a convincing argument. The following theoretical and practical aspects of proof will be discussed: 1. Historical, epistemological and methodological aspects; 2. Empirical research; 3. Recent developments and new ideas in the didactics of proof; 4. Controversial issues. During the working sessions proposals for continuing working and suggestions for research will be discussed.

The Future Scientific Orientation of PME

K. Hart, Chair

Last year members of PME met to discuss the amplification of the constitution which would enable prospective members to accurately judge the aims and interests of the society. We decided upon the information that could be given to participants at ICME and we thought we could profitably continue the discussion on where we are going, at Assisi. The emphasis of the society, as a subgroup of ICME, has always been on the psychological aspects of learning, and to some extent teaching, mathematics. Last year's discussion included suggestions that the psychological aspect should not be the sole focus. The role of teachers in the society was also discussed. It is important that members express their views on these important matters.

Although the present intention is not to provide criteria for the selection of papers, ultimately this will be influenced by the work of the discussion group.

What is Research?

K. Hart, Nuffield Secondary Mathematics, London
O. Figueras, Secion Matematica Educativa, Mexico City

In Paris we had two sessions in which participants discussed what aspects of enquiry might be considered to be essential to "research". A large measure of agreement was reached on some of these. It was felt that we should be sensitive to the methodology employed by other disciplines but we must be certain that our attempts to use their methods stand up to criticism. The group in Paris was organised by a psychologists, two mathematics educators who researched in an "orthodox" way and another who represented countries that were at the start of their establishment of a Mathematics Education discipline in universities. There are still many issues to be discussed and the needs are still apparent.

In Assisi two people will give their own view of the meaning of research. What is 'proper' research? Is it the same as 'good' research? What guidelines are there for research students and those given 'research reports' to critique?

VISUALIZATION IN PROBLEM SOLVING AND LEARNING

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A. Pesci
Dept. of Mathematics
Univ. of Pavia, Italy

The term "to visualize" can be considered synonym of "to see" mentally and can refer both to concrete objects (a chair, a table, ...) and to abstract concepts (peace, hope, a mathematical function, ...).
About the nature of images that the mind forms following a stimulus (visual, auditory or other) many studies, also on contrasting lines, were developed between 1960 and 1980. Following this period the investigations have been orientated on the function that mental images have in cognitive processes.
The influence of mental images on memory tasks, on learning and in particular on the learning of mathematics and problem solving, has been studied.
Regarding problem solving, for instance, the dialectics between external representations such as schemes, tables, arrows, etc. ("diagrammatic models", Fischbein) and internal processes of visualization allows:

- to determine the correct transformations and to foresee the solution;
- to keep in mind, "in parallel", different aspects of the problematic situation;
- to make more transparent the relations involved and favour the correct reconstructions.

The use of mental images varies from person to person: literature distinguishes, more or less decisevly, the "visualizers" from "verbalizers" and "mixers" (Bishop; Clements; Kruteskii). On the other side we could say that today the use of mental images is a "spontaneous" activity, taking into account that in our culture there is not the tradition of this type of education.
Starting from the belief that on one hand the image code is partially autonomous from the verbal one (Paivio) and, on the other hand, that such a code contributes notably to the formation of concepts and to the construction of strategies (Kaufmann), we think that it is essential to solicit and develop in every pupil the capacity to construct and use images: the ability to visualize must be thought of as a basic competence in mathematical activities (Bishop; Lean-Clements) and must be cultivated from the first years of education.
In the frame of these ideas the proposed Discussion Group tends to promote a profitable exchange of experiences with the aim:

- to study what external representations could be appropriate in soliciting in the pupils a rich and functional language of images and what are their "ways" of running;
- to develop tools to analyse the visualization role in problem solving and mathematics learning.

In particular we need several studies in different and specific domains that show the functionality of particular representations in favouring the solution of problems, the acquisition of abilities and the construction itself of concepts.

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COMBINATORICS AND ITS TEACHING: ANALYSIS OF TEACHERS' RESPONSES TO A SURVEY

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SUMMARY

In this work teachers' responses to a survey about Combinatorics and its teaching are analyzed. Participants were 22 in service teachers and 14 trainee teachers who respond to questions concerning actual teaching methodology, suggestions for change, and student's difficulties and interest in Combinatorics.

The answers show that this subject is actually exposed alone, with few connections to other topics, except probability and Newton's formula. Teaching practices are mainly based on the use of the textbook, reinforced with solving complementary problems. Very little use is being done of didactic resources like games and manipulative materials.

Out of the eight topics in the mathematics curriculum for this level (15 years old pupils), Combinatorics is, in the opinion of those teachers, the first subject in order of difficulty and the third for the interest that pupils show in it. The use of Combinatorics in other mathematical subjects and in other sciences is not sufficiently appreciated, especially in the future teachers, who recognized that their training in this subject has been insufficient.

Finally, we present information about the following aspects: content being taught, time spend for it and its planning, types of problems proposed to the students and their relative difficulty in opinion of those teachers and suggested changes for teaching of the subject.

The result of the analysis that we exposed constitutes an approximation to the teacher's conceptions of the subject and also allows us to infer the effect of teaching experience on the same.

This work forms part of the Project PS88-104, granted by Dirección General de Investigación Científica y Técnica. MEC. Madrid.

THE CONTROL IN PROBLEM RESOLUTION

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The context is the situation of the resolution of a problem.
In such a context control is analysed by trying to answer the following questions :

What is it? **When** does it reveal itself? Through **what**?
What is it due to? **What** does it produce? **How** does it act?

The control

is	the action which, during the activity of resolution, links the subjective pole (mental models) and the objective one (grafic models produced by the subject)
it is evident	when the problem is assumed and faced when a strategy is to be elaborated in your first performance when you have to check the final answer when you compare when you doubt when you are stuck ....
it is revealed through	pauses questions gestures further readings of the text different performances ....
it is due to	the iteraction of informations/models/products the conflits of different models the rigidity of standard models and stereotypes the non-adequacy of the performing model the non-completed formation of the performing model obstacles due to previous knowledge ....
it produces	the choice of the starting model the evolution of the model: adjustment/formation the changes in the model the changes in strategy ....
it acts in different ways	explicit / implicit ascending / descending perceptive / notional linguistic / visual ....

MATHEMATICAL OPERATIVITY

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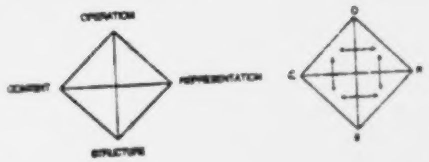
In this poster presentation the relationship between conceptual and symbolic-notational aspects of elementary school mathematics is discussed, and a theoretical framework for discussing didactical aspects of mathematics as a symbol-using discipline is shortly outlined.

The ability to integrate content and form is by many mathematics educators seen as essential for students' understanding of mathematics. The content-form distinction is, however, a tricky one. Understanding of content often means conceptual understanding (seeing relations in and between concepts, i e mathematical structures), which in mathematics is closely related to the representational aspects of concepts. Formal understanding, on the other hand, is often linked to symbolic notation as well as to logical aspects of operations on symbols and their organization.

In teaching experiments it has been observed that a previous routinization of syntactic rules can have a blocking effect on the semantic understanding of these rules. It seems as were the content-form distinction alone an unsufficient tool for analyzing students' ability to work with and understand mathematical symbols. My conclusion is that the didactical value of this distinction may be increased by also taking into account a second distinction, namely that between structure and operation, and by clarifying the role played by logic in this context. Also, a more narrow meaning of form seems necessary for a meaningful discussion of the content-form distinction.

For this purpose the notion of "mathematical form" here is restricted to the arrangement of written mathematical symbols. A "molecular" mathematical form is defined as the spatial relationships between typographical units ("atomic" forms) in written symbolic expressions. "Genetic" forms can be pictured by its corresponding mathematical content, which is not the case for "stipulated" forms. The form of 2+3, e g, can be pictured by •• •••, where the content of the symbolic expression is displayed by *the same* spatial arrangement as that of the symbols.

A unit of analysis (the "crystal") is based on the distinctions between content and form, and between structure and operation. This unit can be viewed from a purely mathematical aspect, or from the psychological aspect of doing mathematics. "Mathematical operativity" relates to the interplay among these four dimensions of mathematics. (See pictures right) An empirical study indicated that logical understanding could be viewed as a mediator for the creation of conceptual-formal links.



KNOWLEDGE AND SITUATED COGNITION: CHILDREN'S KNOWLEDGE OF MONEY IN STORES AND IN SCHOOL

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Differences in learning in and out of school have long been recognized (Resnick1987; Scribner and Cole 1973]. However there has been little consideration of how school practices and everyday practices shape the mathematical knowledge of children who have extensive experience with a topic in both arenas of practice. This study compares what children do with money in schools and stores and describes the development of their money knowledge from preschool through second grade. Specific connections are suggested between practice and knowledge. One sample of children (n=24) was interviewed three times between the beginning of kindergarten and the end of second grade. A second sample of children (n=20) at the same school was interviewed at the beginning and end of their preschool year prior to entering kindergarten. Data about classroom practices are based upon observations done for three years in the classrooms of these two samples. Observations of children's buying behavior were done in five stores twice a week over six months in the neighborhood of the school. It was found that the school based model of money builds from small units to larger (pennies to dollars) within an additive framework in which children are expected to identify the value of collections of coins. It is assumed by teachers that children will know the relationship between different coins and that this will serve as a basis for place value instruction in later grades. In contrast the everyday model of money is centered around dollars and quarters and the main operations are comparison and deconstruction into smaller units. School practices with money include a heavy emphasis upon learning to interpret pictures of money and using money as an exemplar for application of other mathematical skills such as computation and solving word problems. The rare attempts to link money examples in texts to buying contexts were often confusing to the children because of unrealistic prices. Store practices with money include mastery of a 'script' for appropriate shopping behavior and use of the environment as a support device for helping children to making optimal shopping decisions which seemed to exceed their ability to use school arithmetic in the store context. Educational implications of these differences are suggested.

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A STUDY OF MATHEMATICAL REGULARITIES IN GRADE 4

Luciana Bazzini, Department of Mathematics, University of Pavia, Italy

The ability of grasping invariants and of establishing meaningful connections is widely recognized as a fundamental component of mathematical investigation and as a main objective of instruction.

However, it often happens that students find difficulties in discovering hidden equalities in contexts were differences seem to be preeminent. The present study aims at improving the discovery of relationships and rules in the field of numbers and preludes the passage from arithmetics to algebra by focussing on the concept of variable. We are concerned here with the former aspect, which is mainly developed in grade 4 while the latter is defered to grade 5. This study is framed in a wider research on curriculum development in primary school, carried out by the Nucleo di Ricerca Didattica of Pavia. Since primary education, Mathematics is seen with a double face: on the one side it is a source of conceptual instruments aiming at understanding and explaining the real world, on the other side it is a discipline which reflects on itself in order to produce further mathematical knowledge. From the very beginning, the curriculum has given children opportunities to focus on regularities in events, shapes, designs and sets of numbers. The study in question aims at experimenting a supplementary teaching unit on regularities in classes already involved in the project. The content of the teaching unit is exclusively mathematical; pupils are required to undertake an intellectual venture in the world of numbers.

Our hypothesis is that the emphasis on pure mathematical regularities and on their expression by means of a formula is not precocious in the fourth grade. Further, in our view, this can lead to a deeper understanding of invariants, rhytms and structural analogies.

The main foci of the teaching unit in grade 4 are:

- a study of the multiples of a given numbers and the search for a general way to describe them;
- triangular numbers, the sum of the first n natural numbers, the sum of the first n odd numbers and the sum of the first n even numbers;
- numbers sequences to be continued.

This teaching unit was carried out in two classes, which acted as experimental classes, while other two were considered as control group. On the basis of the data we obtained in an initial and in a final test and of the experience we accumulated during the experiment, we can say that the "jump to abstraction" did not cause negative reactions on the part of the pupils. The formulae have been accepted as an extremely concise form of writing. This was mainly due to the teachers' efforts to justify the meaning of those expressions. Many children were stimulated to further investigation in the world of numbers. Even the children who had got the lowest scores in the initial test, improved their performance in the final one.

STRUCTURAL ANALYSIS OF THE ABILITY TO SOLVE THE PROPORTIONALITY OF JUICE-MIXING TASKS: AN INFERENCE NETWORK.

Anne Beland, Robert J. Mislevy, Educational Testing Service, Princeton, New-Jersey.

In cognitive developmental psychology, structural analysis defined the conceptual framework to differentiate the cognitive strategies through adaptive restructuring of task's component relationships at each state of understanding. The adaptive quality of children's strategy choices suggested, however, their interpretation as related to problem structure. The concept of adaptive restructuring was, therefore, described as the process of modifying existing strategies according to the structure of the task. This approach was used to assess the individual's state of learning in the context of proportional reasoning.

Because Bayesian inference networks are a natural extension of model-based test theory, probabilistic inference about cognitive student models provided a framework to understand the individual's knowledge organization. Building upon recent progress in statistics, a Bayesian inference network was constructed to summarize probabilistic relationships among cognitive structure, strategy selection, and correctness of predictions.

Once the individual's explanations were sorted into mutually exclusive and exhaustive categories, each state of understanding was structurally defined as a set of task's components relationships. Then, the learner's state of knowledge was described as the construction of conceptual structures, whose acquisition was defined through the adaptive restructuring of children's strategy choice. To illustrate this approach, simple examples provided by the implementation of an inference network built around this cognitive model will be presented.

POSTER TITLE: Assessing Spatial Visualization: Evidence for Transformed Images
PRESENTER: Dawn Brown and Grayson Wheatley
INSTITUTION: Florida State University, U. S. A.

Previous research (Brown & Wheatley, 1989; Brown & Wheatley, 1990) examining the relationship between spatial visualization and mathematics reasoning has made use of a test of mental rotations (Wheatley Spatial Ability Test, 1978). In order to determine the methods students use in responding to items on the WSAT, a study was designed using clinical interviews. The purpose of this experiment was to determine if students use mental images during this test.

The Wheatley Spatial Ability Test (WSAT) is a 100-item pencil and paper test which can be group administered. In this test a sample figure is shown on the left. Students must then decide if each of five congruent figures are equivalent under rotation . A correction for guessing is applied during the computation of students' scores. Items to which the student fails to respond do not influence the score.

In this experiment the WSAT was administered to four classes of fifth grade pupils in two elementary schools. After the group administration of this test, 10 students were individually interviewed. During these videotaped interviews students were asked how they had answered specific items during the group administration of the WSAT.

In all, WSAT scores were obtained for 85 students in the four classes. As in previous research, individual scores varied over a wide range. Group means for the four classes ranged from 51.3 to 61.7 with standard deviations of about 20 for each class. Reliability coefficients, using Hoyt's ANOVA method (Johnson, 1949), were all .93 and above, indicating a high degree of internal consistency.

Analysis of the video recordings of interviews revealed that each of the 10 students constructed a mental image of one of the figures and rotated that image to make comparisions. Students were also remarkably consistent from item to item in strategies used. Several of the students adopted a "holistic" strategy in which they compared entire figures, while others used only a salient feature. Most students rotated the figures consistently in a clockwise or counterclockwise direction, irrespective of the degree or direction of rotation of the two figures. The time taken to reach a decision and the accuracy of that decision were very different between high and low scorers on the WSAT.

These results lead us to conclude that scores on the WSAT reflect the ability of students to construct and transform mental images. Since students who perform well on the WSAT also perform well on tasks requiring relational understanding of mathematics and nonroutine mathematical problems, it seems likely that imagery is a salient factor in the construction of meaningful mathematics.

PROBLEM SOLVING IN A SOCIAL INTERACTIONIST APPROACH FOR MATHEMATICS TEACHING

ANTONIO CARLOS CARUSO RONCA
MARIA LAURA P. BARBOSA FRANCO
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Based on the evolution of Psychology and its interface with Mathematics teaching in Brazil over the past decades, this study discusses a Mathematics-teaching-oriented proposal for problem solving. In developing our proposal, we conducted an experiment with 50 Math teachers and set up a Math teaching unit for 5th graders. These results as well as that of the teacher improvement program are reported in this paper, and some considerations are made in respect to social interactionalism as a theoretical basis for teaching Mathematics, based on data from the program.

Some conclusions of the use of this Programme of the Improvement of Teachers which has theoretical basis in social interactionalism can be presented:

a) the teacher must develop skills of working with the congnitive structure of the pupil.

b) the knowledge that the pupil has within him is fundamental for the teaching of mathematics.

c) the possibilities of teaching should not be defined according to what the children are able to do on their own.

SYMBOLIC LOGIC - DIFFERENCE OF DIFFICULTIES

Franca Cohen Gottlieb
CEPIM/Universidade Santa Úrsula

1. Introduction:

Logic is one fo the courses in the graduate program (Master Degree) in Mathematical Education at Santa Úrsula University. The work involves bibliografic surveys, individual studies, individual evaluation and seminars with the interpretation by the students of texts touching upo the relationship between symbolic logic and mathematical concepts. In one of these presentation a polemical arose, that was elarged in discussions with the members of the Faculty.

2. The problem:

Let's consider a set $C = \{a,b,c\}$ and the relations S in C: $S = \{(a,c),(b,c)\}$. We define that a relation $R \subset A \times A$ is transitive if and only if the following is true: $\forall x \in A,\ \forall y \in A,\ \forall z \in A,\ (x,y) \in R, (y,z) \in R \Rightarrow (x,z) \in R$. In this case, is the relation S in C, described above, transitive?

By consideration of the logical value of the sentence, we see that S is transitive (we have to examine the 27 possibilities of the logical value of it). Professors and estudents of tradional mathematics or who envisage the subject in a standard form, have difficulties to see the transitivity. They do not understand the situation in wich the premise is untrue. They plead that, in this case, we have to "admit" that the transitity does not hold. Two examples of the situation considered during the discussions, will now be quoted.

a) $C = \{2,3,6\}$ and $S = \{(x,y) \in C \times C \mid x \text{ is a divisor of } y \text{ and } x \neq y\}$

Since traditionally "to be divisor of "denotes a transitive relation, there was no discordance in this case.

b) $C = \{\text{Helen, Betty, Joan}\}$ where Joan is the mother of Helen and Betty.

$S = \{(x,y) \in C \times C \mid x \text{ is daughter of } y\}$

Since tradionally "to be daughter of" does not denote a transitive relation, it was very difficult to make anybody admit that we are in the same case of the first example. We are gratified to observe that, after discussions, our students saw clearly the similitude and understood the right answer.

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THE WORLD OF COMPUTERS - A MENTAL MODEL TO UNDERSTAND ALGEBRA IN SCHOOLMATHEMATICS

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If the difficulties of the pupils in algebra are analysed from a cognitive-science point of view, following conclusion comes up: the weaknesses do not result from the understanding in its contents of mathematical concepts or a lacking skill of dealing with the necessary mathematical tools but a more fundamental deficit have to be postulated casually behind it. The pupils miss a cognitive tool: the ability to represent the functional concept in a functional language and to apply a fundamental understanding of the interaction of syntax and semantics of the functional language.

In the curriculum project for grade 7 and 8 financed by the State of Lower Saxony 1987 until 1993 we reach the understanding of the algebraic part of school mathematics by integrating the algorithmic and axiomatic ways of thinking for pupils. At the moment 18 classes at 6 different schools take part in each grade. The starting point of our project is the analysis of the thinking processes of pupils and not the change of contents of school mathematics. Our own philosophical position sees a duty of the school mathematics to impart to the pupil mathematics as an universal tool for formulating precisely intuitve knowledge and analysing conditions and possibilities of its handling. The analysis of interactions between external forms of representation of mathematical ideas and the resulting internal mental models plays an central figure in this connection. The world of computers is the model world, which *constitutes the meaning*, on the other side it shall *provide* and *practice basic examples* to solve problems when *dealing* with these concepts.

As part of our project we have written textbooks for the pupils and detailed handbooks for the teachers, some of them are translated into English:

COHORS-FRESENBORG, E. / GRIEP, M. / KAUNE, C. : Sätze aus dem Wüstensand und ihre Interpretationen - Ein Textbuch für Schüler zur Einführung in die axiomatische Auffassung von Mathematik. Osnabrück 1990.

COHORS-FRESENBORG, E. / KAUNE, C. / GRIEP, M.: Einführung in die Computerwelt mit Registermaschinen. Textbuch für Schüler, Handbuch für Lehrer. Osnabrück 1990 (English translations available).

COHORS-FRESENBORG, E. / KAUNE, C. / GRIEP, M.: Vertragswerke über den Umgang mit Zahlen. Textbuch für Schüler, Handbuch für Lehrer. Osnabrück 1990.

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The full version of the presented paper is available as Preprint No. 140 in *Osnabrücker Schriften zur Mathematik*. Universität Osnabrück 1991.

Are Lakatosean and Constructivist views of mathematics education consistent?

A. J. (Sandy) Dawson
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Some would argue that one's philosophical and psychological positions vis-a-vis the teaching and learning of mathematics need not be consistent, that one does not, for example, have to adopt a constructivist position philosophically in order to function pedagogically in a constructivist fashion. Kloster and Dawson, after a review of positions held by a representative selection of mathematics educators in North America during the past five decades, found that though some sought consistency between their philosophical and psychological perspectives, currently many *adopt* a constructivist orientation yet claim that children *must learn the right mathematics*, as if constructivists would agree there is any *right* knowledge. This does not seem particularly troublesome to these individuals. They seem to believe they have a more-or-less complete and functional approach to the teaching of mathematics. Still, to play a bit on Gödel's results, one can have a complete system only at the price of consistency. If one has a consistent system, the price of course is that it is necessarily incomplete. I personally find it easier to live with incompleteness than with inconsistency. Until very recently, fallibilism and constructivism would appear to have been competing epistemologies. However, von Glasersfeld, partially in response to Noddings, has adopted the stance that constructivism is a post-epistemological position. Noddings describes constructivism as a cognitive position and methodological perspective. Constructivism is the view that learners must of necessity make sense of their own world, that one's view of the world is created within one's self and validated with reference to the culture within which one functions. Fallibilism, as applied by Lakatos to the growth of knowledge in mathematics, is an epistemological stance. In this view, mathematical knowledge grows by a process of proof and refutation, and mathematical knowledge is not independent of a knower. Mathematicians are meaning-makers, and their criterion for what is acceptable mathematics is that it has not been refuted. A mathematical proposition is viable in this view if no counter-example has been found to it and/or a *proof* for it can or has been constructed. What constitutes acceptable proofs and counter-examples is negotiated within the mathematics community. At a surface level, then, fallibilism and constructivism offer the hope for the creation of a consistent perspective on the philosophical and psychological foundations for mathematics education, and the poster session will provide opportunities to discuss that proposition further.

INTRODUCTORY ALGEBRA REVISITED BY UNIVERSITY STUDENTS

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Many university students enroll in an introductory algebra course, even though they have completed equivalent study in secondary school. A point of view of learning at this level has been developed, with consideration of faculty and student perceptions of "Mathematical Literacy".

Basic Characteristics of Mathematical Literacy

Faculty Perspectives Include:

- Some understanding of the cultural history of mathematics
- Ability to use hand calculators and, ideally, a computer
- Solving an algebraic equation to make a routine decision
- Solving systems of linear equations and graphing them
- Reasoning, using numerical or measure relationships
- Computation skills involving percents, decimals, fractions
- Underlying facility with basic operations using functions
- Setting up simple linear math models
- Using combinations and permutations, and using proportions
- Understanding elementary probability and statistics

Student Perspectives Include:

- Analyzing quantitative problems, step by step
- Solving complex numerical problems
- Thinking rationally to solve everyday problems
- Using logical decision making processes
- Finding solutions using algorithms
- Observing patterns in a series of numbers
- Collecting and analyzing data
- Using graphs and charts
- Working with computers
- Reducing math anxiety

A Point of View about Learning and Involvement in a Review Course

Students should be active partners in their learning.
A "mature student" is ready to participate in the learning process.
A review course should help a student reach this maturity.
Patterns and relationships provide insight into fundamental skills.
Checking one's solutions keeps the problem intent in focus.
Interesting presentation must be blended with instructive experiences.
There is a great variety among students, so a course should accommodate a range of abilities, needs, and different modes of learning.

GENDER-RELATED DIFFERENCES IN HIGH SCHOOL MATHEMATICS PERFORMANCE AND MATHEMATICS LEARNING STYLES

Carol J. Fry
California State University, Fresno

Although females tend to earn higher grades in mathematics than do males, they often score lower than do males on standardized tests of mathematical performance. Kimball (1989) has proposed that this discrepancy may be attributable in part to gender-related differences in learning styles.

The present study was designed to examine the relationship among the following variables:

- Classroom Performance in Mathematics (Grades in Algebra I, Geometry, and Algebra II),
- Preliminary Scholastic Aptitude Test – Mathematics (PSAT-M) Scores
- Learning Styles and Interests (Mathematical Confidence, Technical Orientation, and Desire for Structure), and
- Sex of the Student

The subjects were 397 Algebra II students (183 males and 214 females) from six high schools in the Columbus, Ohio (USA), city school district. The *Mathematics Learner Profile* (Fry, 1988) was administered during the week preceding final exams. High school mathematics grades and PSAT-M scores were obtained from the students' permanent records or from their classroom teachers.

Compared with males, females earned proportionally more of the higher grades (A's and B's) in all three classes. Although mathematical confidence was significantly related to course grades, females at each level of classroom performance expressed less confidence in their ability to do well in abstract mathematics or to perform well on problems requiring cognitive restructuring or mathematical applications. Females also reported significantly less interest in technical hobbies or careers than did males, and expressed a greater desire for structure when learning mathematics. Scores on the PSAT-M were significantly related to grades in Geometry but not to grades in either Algebra I or Algebra II. Females performed significantly lower than males on the PSAT-M (Males: $M = 50.0, SD = 11.5$; Females: $M = 44.0, SD = 9.3$).

The findings suggest that females, to a greater extent than males, suffer the consequences of traditional instruction and assessment practices in mathematics. Proficiency in the rote memorization of algorithms may result in short-term success (high test grades), but this success is perhaps achieved at the expense of broader, more long-term goals – conceptual understanding and the development of confidence and competence in mathematical problem-solving and reasoning skills.

GROUP CASE STUDIES OF SECOND GRADERS INVENTING MULTIDIGIT ADDITION PROCEDURES FOR BASE-TEN BLOCKS AND WRITTEN MARKS

Karen C. Fuson and Birch Burghardt
Northwestern University

Small groups of second graders added horizontally presented 3- and 4-digit numbers using base-ten blocks and written marks recordings of the block procedures over a five- to eight-day learning situation. Sessions were videotaped. All children displayed increased conceptual understanding of place value and multidigit addition (the theoretical framework described in Fuson, 1990, provided the framework for the presession and postsession interviews and analysis of the videotapes), and most children demonstrated better written addition competence. The six groups displayed individual patterns of invention and learning that were dependent upon the personalities and mathematical understandings of the group members. Children easily added with the blocks, devising accurate strategies for multiunit sums of ten or more (e.g., twelve tens, sixteen ones, eleven hundreds). Many children did not spontaneously link the block addition to marks addition, instead operating in two separate worlds. When experimenters intervened to have children link blocks addition to marks addition, the blocks were a powerful support for conceptual understanding of marks addition and children self-corrected incorrect invented procedures. Blocks words were in some cases a more powerful support than were English words, and complete verbalization of trading seemed to be very helpful in facilitating understanding.

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THE DEVELOPMENT OF A STUDENT THROUGH MATHEMATICAL TEXTBOOKS

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Tomsk Pedagogical Institute, Tomsk, USSR

In the process of teaching mathematics we are confronted with a problem: how to make the logics of a text-book correspond the students' psychological peculiarities and provide their intellectual development. We shall dwell upon the main demands which are realized in a series of text-books on mathematics for children of 11-15.

1. To create such logical structure of problems, which corresponds to the components of conceptual thinking. Every problem must fulfil its psychological function: to transfer the contents of notions into the language of images; to teach to single out the characteristic features of notions and connections between them; to link the notions inside mathematics and outside it. We try to make pupils look upon mathematics by eyes of a physicist, economist and so on. This gives them the possibility to understand the mechanisms of appearance and usage of mathematical notions.
2. To form both the special mathematical and common skills: planning, controlling, formulating a problem, making up hypothesis and so on. The pupil can choose the manner of study of theory; the level of training; the manner of control and checking. The advanced children are provided with special talks.
3. To give not only knowledge but meta-knowledge; the pupil gets acquainted with different information about the ways of cognition of the world and about himself.
4. The manuals are written in lively dialogical form of a fairy-tale or a science-fiction or a play or a talk with the authors. These methods are not simply entertaining but teach kindness, mutual understanding, proper style of human relations.

COGNITIVE THINKING LEVEL ON FRACTIONS WITH 8th GRADE STUDENTS

GIMENEZ , Joaquim

Dept. Educació i Psicologia . Univ. Barcelona . Div. VII. SPAIN

Aims of the study .

Our previous studies (Giménez 1989) showed different factor elements in a rational numbers thinking test . Also recent research (Behr & Harel 1990) draw a picture over semantical analysis on rational numbers thinking.

Many people says that it's very difficult to accept radical changes in teacher training processes, but it's posssible to have a positive intervention including elements suddendly forgotten and promote some cognitive changes over cognitive level (as in Bye et al. 1981).

We showed , in a curricular perspective in Spain, that there are points-of-break (Giménez 1991) . One of them there were that theachers did'nt accept the existence of a low cognitive level on the subject with 8th grade students. Therefore we tried to prepare an instructional model based on Kieren's perspective about building intuitional knowledge , in order to promote a high cognitive level.

Experimental characteristics .

Three teacher and 135 students aged 13-14 years old on suburban schools near Barcelona were involved in the experiment. Five groups were tested before and after the experience. All the groups had 25 lessons of about 50 minutes on the topic. We used an enlarged version of RTT Kieren test with 33 items to assess the cognitive level changes. We also assign three levels according the Calgary Study (Bye et al. 1981) .

Some results.

The cognitive level in experimental groups was meaningful better than control groups as can be seen from the average of children in each level. We can also observe different results on different constructs. We also observed many changes to better strategies than initial ones in experimental groups.

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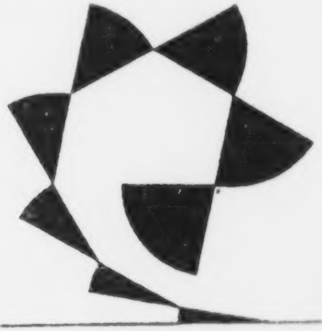
MATHEMATICS AND MODERN ART

Ulrich GREVSMÜHL,
Pädagogische Hochschule Freiburg, Germany

The international movement of constructive art, which originated at the beginning of the 20th century in the De Stijl movement in Holland, in the German Bauhaus and in the East European constructivism, has made it as their aim to investigate the principles of orders and structures in nature, in our society and in our technological environment and to express these in visual form.



NORMAN DILWORTH
Born in Wigan, 1931. Lives in Amsterdam and Paris

Parts of a circle
(1986, hardboard painted green, 211 cm high)

The works of constructive art are of special interest for the teaching and learning of mathematics because of their mathematical, perceptional and conceptional implications. Many of the works may be used as starting points for mathematical investigations and problem solving activities and give the opportunity to experience art from the rational side.

In close cooperation with many artists in Europe and overseas the author has collected a large number of mathematical problems based on works of art. In the poster session a selection of these problems will be presented in form of worksheets for students at various levels.

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TEACHING MATHEMATICS IN NATURAL LANGUAGE

Eyn-Ya Gura
The Hebrew University, Jerusalem

To the question that was presented by Davis (1987): "What Kinds of Experience Should We Provide for Students in Order to Help Them to Learn Mathematics?", my answer is: Let us try to teach them mathematics in everyday language, which means teaching mathematics using as little of the remote and alienating formalistic language as we can.

In order to teach "real mathematics" we need to introduce students to as much mathematical content and as many ways of presenting that content as possible. One of the ways that is not sufficiently familiar to students is mathematical argumentation via everyday language. We must try to introduce the students to a mathematics which is essentially closer to human experience, where it will be legitimate to use everyday language.

Having all this in mind I created a textbook in game theory for high-school or equivalent level. Game theory was chosen because "the language of game theory - coalitions, payoffs, markets, votes - suggests that it is not a branch of abstract mathematics; that it is motivated by and related to the world around us; and that it should be able to tell us something about the world" (Aumann, 1985). Yet game theory is a branch of mathematics and its medium is mathematical model.

As a result of teaching a course in game theory which emphasizes concepts and ways of thinking we found that it contributes to an improved acquaintance with mathematical concepts. We found that we were teaching "real mathematics".

LEARNING BY REPRESENTING KNOWLEDGE TO OTHERS: DESIGN AS A MATHEMATICAL LEARNING ENVIRONMENT

Dr. Idit Harel
MIT Media Laboratory, Epistemology & Learning Group

This presentation describes how we implemented and studied a model for the *constructionist vision of computer programming as a source of learning power.* (On "Constructionism" see: Papert 1990, Introduction to Constructionist Learning. In Harel, I. (ed.) *Constructionist Learning: A 5th Anniversary Collection.* Cambridge, MA: MIT Media Lab; and Papert 1991, Situating Constructionism: A Theoretical and Social Context. In Harel & Papert (eds.) *Constructionism.* Norwood, NJ: Ablex.) In general, this model project presents a shift in research methodology and educational practice by casting learners in the role of instructors and knowledge communicators rather than information recipients, and in the role of media producers rather than consumers. We conduct our studies within a context of a computer-rich culture in an inner-city school in Boston. By implementing mathematical "software design projects" with elementary-school children (ages 9-11), we explore how learners can attain new levels of insight when they develop complex mathematical software products for other students in their school.

At the PME-91 Conference, by using posters and videos, I will demonstrate the lively and challenging implementation and analysis of this extended notion of children's programming and mathematics education; in particular, the ways in which design, productive activity, social context, and technical contents are essential to mathematical understanding. It was found that the young designers learned not only about mathematics (rational-number concepts) and programming (Logo), but also about instructional design and user interfaces, representational, pedagogical, and communicational issues—presenting a new paradigm for mathematical computer-based activities in schools. For more information on methodology (procedure, data collection and analysis) and results (quantitative, statistical, comparative, qualitative, and case studies) see:

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A COMPUTER ENVIRONMENT FOR SIMULATING MECHANICS

Sara Hennessy,
Malcolm Byard, Stephen Draper, Rosalind Driver, Roger Hartley, Conroy Mallen,
Ramzan Mohamed, Claire O'Malley, Tim O'Shea, Eileen Scanlon, Daryll Twigger

Open University, Leeds University and Glasgow University, U.K.

The aim of our research is to clarify and describe the process of change in learners' conceptual understandings of natural phenomena, primarily in the domain of force and motion. We have devised a range of intervention strategies to promote conceptual change, focusing on the development of computer software for exploring, modelling and challenging children's reasoning.

Previous work shows that school children, in seeking to understand and explain phenomena they experience in the physical world, construct mental models of how that world behaves. These models are used to provide explanations that seem to fit with the learner's experiences: they are very persistent and are not normally changed by conventional teaching. In order to help children to reflect critically on their conceptions, and to modify them, we have implemented a software environment called DM^3 (Direct Manipulation of Mechanical Microworlds). This environment has been used to create an ordered series of interactive simulations of motion under forces. These simulations allow pupils to observe and manipulate, both informally and in a structured experimental manner, the behaviour of objects in motion. The sequence comprises four kinds of scenarios, including horizontal motion (a) with friction ('Cardboard Box'), (b) without friction ('Rocket Skater') and (c) with speed-dependent resistance ('Speed Boat'), followed by (d) vertical motion under gravity ('Parachutist'). These scenarios enable children to collect empirical data about prototypical motions and to identify simple lawful relationships between variables in the computer world. They make clearer the Newtonian rules operating.

The simulations are not designed to stand alone but to constitute key components in an integrated teaching package which includes a range of carefully prepared support materials. Both our computer-based and practical activities involve pupils in making predictions about outcomes, providing reasons for them, and then testing their predictions. Thus, pupils carry out experiments either within a simulation or with real equipment and then attempt to explain any differences arising. Our activities do not directly challenge pupils' prior conceptions, but aim to allow them to construct an alternative articulated set of rules which have internal coherence. These rules can be used to predict and produce particular motions within the scenarios, and consequently to describe and explain 'real world' behaviour.

In constructing and testing our computer simulations and other activities, we have undertaken extensive empirical work with pairs, small groups and currently, a whole class of pupils. We are carrying out this large-scale classroom intervention to assess the impact of our sequence of scenarios, when supported by practical and written work. A class of 29 children aged 12-13 are being taught mechanics (by their own teacher) using our new curriculum over a period of 7 weeks. Preliminary results of this empirical work will be presented.

THE ATTITUDES OF STANDARD EIGHT PUPILS TO MATHEMATICS IN THE ALICE-MIDDLEDRIFT CIRCUITS IN CISKEI-SOUTH AFRICA

Yaliwe Jiya University of Fort Hare South Africa

"A positive attitude is an important school outcome in itself." (Haladyna. 1983)
The study being reported on here, was designed to tap the attitudes of post-primary school pupils at the standard 8 level. Approximately 600 pupils were administered an attitude questionnaire to. 532 were returned by 9 schools (7 schools in the Alice Circuit and 2 in the Middledrift Circuit). The 2 major groups of respondents consisted of 5 schools for pure maths and 4 for functional maths. There was great consistency in patterns of attitudes between the 2 groups. This is a report on the research and the findings.

INTRODUCTION

While the measurement of school achievement has had a comparatively long history, the systematic measuring of pupil attitudes toward school and various subject matter comprising its curriculum, has been conspicuously absent in Black Education in South Africa. Reports have been written about poor performances of pupils, especially in science and mathematics. Unfavourable statistics on teacher underqualification in these areas, have also been given, and so have unfavourable statistics on the teacher:pupil ratio been given. Poor results have been attributed, in some quarters, to the "culturally deprived environment" from which Blacks generally come. (South African Institute of Race Relations. 1984:438). All these have been concern for worry for as long as I can remember. And, I believe, some attempts are being made to remedy these. Yet no attempts have been made to measure the pupils' attitudes toward mathematics and science, the most failing subjects.

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GRADUATE COURSE IN MATHEMATICAL EDUCATION

OPERATION AND STRUCTURE

Estela Kaufman Fainguelernt
GEPEM/Universidade Santa Úrsula

1. Introduction:

Master Degree in Mathematical Education aroused from the verfication that Graduate Courses in Pure Mathematics did not fulfill the education of Mathematics teachers regarding different approaches of teaching and methodologies.On the other hand,graduate courses in the area of Education had gaps to be filled with the specific themes of mathematics. The teacher of Mathematics hasto have proficiency in specific knowledes and,simultaneously, capacity of developing different approaches of the same subject.The Master's Program at Santa Úrsula University began in April 1989.Currently,it has enrolled 37 students:14from the first year, 13 from the second and 7 beginning this year. A modification of preliminary exams caused a decreasing in the number of approved candidates: now there exists a dissertation value 40% of the maximum to be obtained and problems of the real world completing the remaining 60%.Since 1990, each student has an academic advisor for planning their studies,each professor with at maximum four students.

2. A new structure of disciplines:

The Master Program in Mathematica Education has only one area of research, and one of the conditions for the student to earn the degree is to be approved in at least 30 credits. The distribution of the disciplines along the course was revised,and now it is the following: Fundamental, Basic Integration, Educative Communication and Mathematics. Comparing the former structure with the current, it can be verified the inclusion of a nucleus called Fundamental, covering the aspects of Fundamental Ideas of Mathematics, Methodology of Research in Mathematical Education, and Transdisciplinarity of Research.

3. Conclusion:

As a result, the implemented modification resulted that the students in thesis now are disseminating this field of the knwledge,considered new in Brazil.The students,who are active teachers,are filling coordination positions in private and public schools,and their papers are being accepted in national an international congresses.

Towards the meaningful teaching and learning of mathematical concepts and algorithms: the third teaching experiment

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Department of Teacher Training in Oulu
University of Oulu

The current study forms one part of the larger project with the practical aim of promoting the utilization of the pupils' solution methods and real-world experiences in the school mathematics. Learning and teaching of ratio and proportion was chosen as the speaific target for research, because a good deal of previous knowledge on the development of proportional reasoning was available.

The main purpose of the third teaching experiment was to get systematical knowledge of the effects of both contextual and conventional teaching programs on the mental and written solution processes of 8th-grade pupils (aged 14+, n=113) in problems that require proportional reasoning. It was also examined to which extent the custom-made computer test is applicable and reliable in the assessment of the development of proportional reasoning. All the 8th-grade pupils of the Teacher Training School in Oulu and their mathematics teacher (n=113, n=8) took part in the study. Each one of the teachers taught his own class and the author taught the test group (10 lessons & the written final exam & initial, final test computer based test "Juice").

The experiments and results that were achieved are similar to those obtained in the first two experiments of the project. A teaching of mathematics that starts from real-world problems (liquid mixtures, geometrical similarities) and ends up in their valid solution is experienced as interesting and useful. Both this teaching method and the text-book method had a similar effect on the development of proportional reasoning. Development was most obvious in the highest grade category in which most of the pupils (83%) reached the reasoning levels corresponding the stages of formal operations. The development of proportional reasoning was weakest in the grade category 7-8. In the written problems most of the successful solutions were based on the formation and solving of proportion. In that way many of the pupils who only reached level E (a:a vs. b:b) in the final mental test "Juice" solved the missing-value problems which were logically much more demanding. The computer based test "Juice" proved to be applicable and reliable.

REFLECTION IN MATHEMATICAL PROBLEM SOLVING AND DESIGN

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L.L. Khisty, University of Illinois at Chicago, Chicago, Illinois, USA

This paper describes and discusses how the principles enunciated by Schon in the Reflective Practitioner (1983) and Educating the Reflective Practitioner (1987) were applied in teaching a junior-level course in mathematical problem-solving and design. The main objective in teaching students to be reflective was to develop higher cognitive skills to enable them to be independent and creative problem-solvers. By providing students with an active learning environment it was possible to introduce them to the control and regulation aspect of metacognition.

The paper begins by describing the problem-solving/design process and then goes on to show how Schon's concepts were applied in class through twelve techniques for encouraging "reflection." Observations and evaluations made by students along with those of the instructors are recorded. The results of this study demonstrate how stereotyped classroom instruction can be modified to develop reflection-in-action and thus enhance the problem-solving abilities of students.

Schon's concepts of reflection-in-action were kept alive in the process of educating several batches of students to be independent learners and creative problem-solvers. The skills needed to apply Schon's ideas were by no means easy. Indeed, it was quite a challenge to motivate students to "learn how to learn" at practically every stage. Probably, the most difficult part was in descending from the hard, high ground of the theorist to the messy, boggy swamp below of the practitioner, but the large majority of students emerged alive with enthusiasm and confidence.

THE CONSTRUCTION ZONE IN MATHEMATICS TEACHING WITH HISPANIC STUDENTS

Lena Licon Khisty, University of Illinois at Chicago

This study examines the sociolinguistic nature of teachers' interactions with Hispanic students and its role in the development of mathematical cognition. Assumptions regarding acceptable mathematics teaching recently have changed dramatically; we now recognize the important role interaction plays in the learning process. However, Vygotsky (1978) and others (Newman, Griffin, and Cole, 1989; and Tharp and Gallimore, 1990) propose that a key element in the learning interaction is the assisting performance of an "enabling other" who provides the learner with a "scaffold" via dialogue and a model of the intended outcome.

This microethnographical study is part of a larger project sponsored by the National Science Foundation to qualitatively examine the nature of mathematics teaching with high risk bilingual students in primary grades. Data consist of field observations and video tapes gathered over a year. Three teachers who provide a contrast in their interactions with students are discussed in order to delineate assisting behaviors. Two teachers demonstrate deliberately weaving mathematics into all activities, creating situations where students need to use mathematical words in their first and second language, and telling mathematical conceptual stories. However, the third teacher makes evident that new practices and reorganizations of classrooms can be fruitless without consideration of appropriate instructional conversation.

Poster Presentation :- **Mathematics of the Bushmen**
Presenter :- Hilda Lea
Institution :- University of Botswana.

A good example of mathematical ideas used before recorded history can be seen today in the daily activities of the Bushmen, as their way of life has remained unchanged for thousands of years.

In the Central Kalahari Game Reserve in Botswana there are a few thousand Bushmen. They are the only people allowed to hunt there, but they must hunt of foot and use bows and arrows. They are self sufficient, find their own food, and make their own clothing, tools and weapons. They live in a harsh environment yet they have worked out a satisfactory way of life. Government policy in Botswana is to integrate the Bushmen into the wider society of the country.

Counting. Two count system, one, two, two-one, two-two; hand is the word for 5.

Distance. Precise locations required to find water, seasonal food and honey; how far away is the game; animals are identified near the fire at night by the height and colour of eyes; lion's roar has a lower note when near.

Time. Words for different times of day and night; months measured by moons and years by passing winters; babies spaced by 3-4 years; time estimated by shadows.

Direction. Herds come from the same direction as migrating butterflies; bee flies in straight line for honey; wind direction is important.

Classification. Plants classified according to growth habits and whether edible or medicinal; animals classified into those which can be eaten, and those which harm, bite or sting.

Spatial ability. Has sense of position and direction and can constantly update mental map; precise locality of plant known even though many kilometres away.

Tracking. Can identify footprint of an individual animal or person; can deduce species, male or female, speed, and when an animal passed.

Technology. An effective and light weight effective tool kit; bow made from raisin bush with stress points bound with sinew; quiver made from root bark; arrows have range 25-100 yards; string made from fibres of a leaf.

A sip well shows capillarity.

Craft. Jewellery made from ostrich egg shell; skins cured from bark and roots; dance rattles are cocoons of moths with shell inside; thumb piano has wooden base with metal keys.

MOTIVATION À L'APPRENTISSAGE DES MATHÉMATIQUES CHEZ DES ADOLESCENTS MONTRÉALAIS DE SECONDAIRE I À SECONDAIRE IV

Nicole Mailloux, Ph.D.
Université du Québec à Hull

Dans notre recherche nous avons traité de l'un des facteurs de l'apprentissage des mathématiques: la motivation. Nous nous demandions si les élèves étaient motivés en mathématiques au premier cycle du cours secondaire et si leur degré de motivation changeait au deuxième cycle; s'il y avait correspondance entre leur motivation en mathématiques et quatre autres variables à savoir: la perception qu'ils ont d'eux-mêmes en mathématiques, leur attitude envers leur professeur de mathématiques, la perception qu'ils ont du besoin des mathématiques et leur succès en mathématiques. Nous espérions pouvoir tracer le profil de l'élève motivé et de l'élève non motivé afin de proposer des pistes pour solutionner la non motivation des élèves en mathématiques.

Un questionnaire écrit de type Osgood et comprenant trois variables: le cours de mathématiques, l'élève en mathématiques et le professeur de mathématiques a été construit pour vérifier la relation entre la première variable et les deux autres, de même qu'entre cette première variables et d'une part la perception du besoin des mathématiques et d'autre part le succès en mathématiques.

Les résultats obtenus ont montré que les élèves étaient motivés aux deux cycles du Secondaire. Nous avons cependant isolé quatre groupes extrêmes, à savoir les élèves très motivés qui restent les plus motivés, les non motivés qui restent non motivés, ceux dont le degré de motivation baisse le plus et ceux dont le degré de motivation augmente le plus.

Ces différents groupes nous ont permis de montrer qu'il y avait correspondance entre d'une part la motivation en mathématiques et d'autre part soit la perception de soi, soit le sentiment de besoin des mathématiques, soit le succès en soi, soit le sentiment de besoin des mathématiques, soit le succès en mathématique.

Nos conclusions rejoignent ainsi celles de Reyes (1984) qui prétend que la confiance en soi, le succès et la perception du besoin vont de pair avec l'apprentissage des mathématiques et qui souligne l'importance de l'influence du professeur au niveau de ces variables.

THE AGAM METHOD OF VISUAL THINKING

Zvia Markovits, Rina Hershkowitz and Bat-Sheva Eylon
The Weizmann Institute of Science, Rehovot, Israel

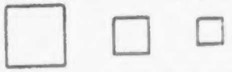
The Agam method of visual thinking educates children to understand and absorb the world of shapes around them, to know its basic elements, to control this shape world by having the ability to change it, and to develop creativity.

For visual thinking we need to learn some "visual language", which is basically similar to learning any other language, and so the method provides the children with some visual "ABC", and then with "words", "sentences", etc.

The visual thinking education method was first created by the artist Iaacov Agam for children from preschool through the first grades of elementary school. The program was redeveloped, implemented and followed by careful research, by a project team of the Science Teaching Department at the Weizmann Institute of Science, for about seven years.

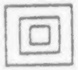
One of the tests the children were given in the research was designed to test visual abilities. It was administered to 53 fourth grade students who participated in he program for several years. The following is an example of an item:

Draw different combinations with the three squares

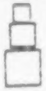


(16 spaces were given.)

All students gave at least four combinations, and 30% of the students gave 16 combinations. In all, 630 combinations were given, 11.9 combinations per student. The following were the most "popular" combinations. The number below each combination represents the number of times the combination was given.



a) 73



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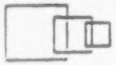
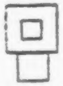
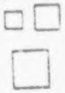


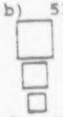
c) 38



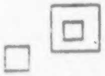
d) 29



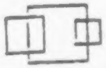
e) 27



f) 25



g) 24



h) 24

Combinations a) and c) were shown to the students as examples of possible combinations when given the task, and this explains their popularity. Combination b) was popular probably due to its similarity to the toy towers children use to build with cubes. In about two thirds of the 630 combinations, all three squares share the same relationship (for example in c) the relationship is intersection), and only in one third the squares share more than one relationship (as in d) and g)).

LENA'S BELIEFS AND ATTITUDES TOWARD MATHEMATICS AND MATHEMATICS LEARNING IN THE CONTEXT OF MATHEMATICAL LOGO BASED ACTIVITIES*

João Filipe Matos
Departamento de Educação Faculdade de Ciências
Universidade de Lisboa

Summary

Since the early sixties, different theoretical perspectives have conducted to different paradigms of research on attitudes toward mathematics. More recently, some researchers shifted the focus of the study to the affective factors embedded in the learning of mathematics – particularly on problem solving.

In the search for factors that help the characterization of the nature of students' attitudes toward mathematics, it is adopted in this study that all experience is mediated by interpretation. Therefore, the evaluation of students' attitudes results from the identification and understanding of beliefs through which their representation of mathematics and mathematics learning is built upon.

This paper reports a case study of an 8th grade student who developed for one school year several mathematical Logo based projects and investigations. These activities took place in the school computer center, in a time slot of two hours / week under the supervision of the mathematics teacher. Data was collected from (a) observation and video recording of her work and interaction with pairs and teacher, (b) semi-structured interviews in three moments of the study, and (c) interview of her pairs, parents and teachers. Content analysis of this material was carried out in three phases (pre-analysis, coding and inference).

Results of this study point to the conclusion that the work developed with computers (a) slightly influenced Lena interpretation of difficulties, suggesting that difficulty is accepted as something that is part of mathematical activity and not necessarily an undesirable element, (b) favored the emergence of a feeling of control over the problem situations, and (c) contributed to the belief that learning mathematics involves a progressive construction of meanings and ideas replacing the 0-1 understanding state held by Lena in the beginning of the study.

Nevertheless, Lena central beliefs on the idea that the fundamental goal of mathematical activity is to solve exercises and problems to achieve good scores in tests didn't change her attitude toward mathematics.

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* The study reported in this paper was supported by the Instituto Nacional de Investigação Científica and Project MINERVA.

A METHOD OF RESEARCH

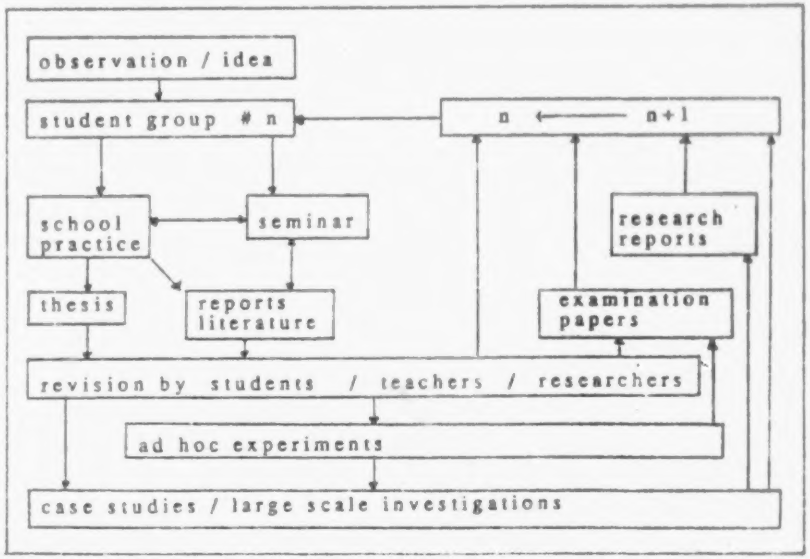
Hartwig Meissner
Westfälische Wilhelms-Universität Münster
Federal Republic od Germany

This paper does not fit into the (well described, but narrow) frame of a pure "research report", because we report how we do research. Those who expect research results may not be disappointed: We also describe results of the research we did in the topic areas geometry, impact of calculators and computers, early number concept, and problem solving. Our concept:

(1) Research without funds for the salary of research assistants or specialists (by cooperation with teachers and teacher students).

(2) Research in action. Researchers are learners and learners are researchers. Thus our teacher training institute combines preservice, inservice and research questions to form working groups of university staff members, teachers and teacher students.

(3) Step-by-step research. At the beginning there is an observation or an idea. A first group (#1) works on it, in schools or in a seminar. The output is forwarded to the next group (n+1 → n). The quality level for a specific topics raises from "loop" to "loop". Each group gets some more results and finally several "generations" of working groups have produced new results on a scientific level:



THE ROLE OF PERCEPTIVE ASPECTS IN GEOMETRICAL REASONING :
a clinical approach with junior high-school pupils *

Ana L. MESQUITA
Lisbon University

The purpose of this poster is to examine, in a gestalt approach, the influence of the perceptive aspects in geometrical reasoning. In the setting of clinical interviews, eleven pairs of pupils work on a simplification of a Euclid proposition (prop. XLIII, Book I). Our study suggests that the children's geometrical reasoning is strongly influenced by their own apprehension of the figure. As a trend, a global apprehension (in which the figure is perceived as a division of the whole) privileges subtractive treatments and direct reasoning. An analytic approach (in which the figure is perceived as a recomposition of its internal elements) is associated with additive treatments and indirect forms of reasoning, such as backwards reasoning and reasoning by absurd.

The poster is abundantly illustrated by fragments of pupils' speechs.

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* This research had been done with a Gulbenkian Foundation grant.

HELPING SECONDARY SCHOOL STUDENTS TO UPGRADE THEIR VAN HIELE LEVELS

LILIAN NASSER
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO (BRAZIL)

During the 1990 school year, a teaching experiment has been carried out in state schools of Rio de Janeiro, Brazil, involving 340 students of 7th grade (13-14 year-olds) with the following purposes:

(a) to investigate the causes of the difficulties encountered by students in Geometry at this grade;

(b) to verify if these difficulties can be alleviated by appropriate instruction, based on the van Hiele Model of Thinking;

(c) to check if it is possible to upgrade the students' van Hiele levels in the topic of plane shapes (assingned at the beginning of the year), through special activities;

(d) to compare the performance in the topic of "congruence" of two groups: an experimental group, having a Geometry course based on the van Hiele theory of thinking, and a control group, having a traditional Geometry course.

This poster concerns the results of item (c) above, comparing the van Hiele levels attained at the beginning of the school-year to the ones attained at the end of year, after special instruction. The activities designed for this purpose will be shown, involving the handling of cutouts, classification of shapes, listing and comparing properties of quadrilaterals, leading to the inclusion of classes. The tests used to identify each student's van Hiele level will also be shown.

Study on Effects of Three Teaching Trials with Computational Alternatives in Mathematical Problem Solving Context

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Katuhiko Shimizu, National Institute for Educational Research
Yasuhisa Hirai, Institute of Education, University of Okayama
Robert Reys, College of Education, University of Missouri
Barbara J. Reys, College of Education, University of Missouri

Throughout the thirty years period since 1958, paper-pencil computation has been highly emphasized in the elementary school mathematics curriculum in Japan. Computational estimations are emphasized in the 1989 latest version of the Ministry of Educational curriculum recommendation. Solution by paper-pencil has been used to solve problems in the classroom activities. We would like to assert some typical distinctions of an effectual outcomes among three trials to use computational alternatives, namely, estimation, pocket calculator and paper-pencil. Using estimation and pocket calculator seem to be both more efficient ways than paper-pencil in mathematical problem solving. This paper reports a comparison of effectual outcomes to teaching trials with these three computational alternatives.

We conclude to assert the value in the use of estimation and pocket calculator in the process of understanding, solving and retension of the effect. This research suggests that higher-order thinking of mathematical problem solving is developed by the use of computational estimations especially. It is useful for students to replace given number with more simple number and to change to a more simple situation. The value in the use of estimation in process of understanding, solving and extending the problem lies in its usefulness for the students to get the structure of the problem more easily. Also, it is useful for the students in the forecast of solutions and to make similar and general problems.

TITLE: Misconceptions in Probability: a comparison of gamblers
and non-gamblers within the Australian Social
Context:Poster presentation summary.

AUTHOR:Robert Peard

INSTITUTION:Queensland University of Technology

The results of research into misconceptions in probability are presented as part of a larger study involving cognition of probabilistic ideas within a social context. Preliminary research results were presented at PME-14, Mexico 1990 .The widespread occurrence of the phenomenon of gambling within the Australian social context has lead to the question "does gambling constitute a form of ethno-mathematics with Australian Society?". (Peard, 1990a). Haig (1985) reports that Australians spend more per capita on legal gambling than any other country, amounting to approximately $2000 per working adult. When expressed in terms of percentage of expendable income, this figure is double the world average (p.67). Haig reports that 80% of this gambling is related to track activities.This paper examines the perception of probability by pupils whose social background includes gambling at the track in comparison with non-gamblers.

Conclusions

The use of availability and representativeness is not as common in our sample as the literature would suggest and further research is necessary. However, it would appear that those who participate in games of chance involving cards and dice and have some knowledge of gambling make very little use of either availability or representativeness.Misconceptions regarding the concept of fairness are common. Non-gamblers have little concept of mathematical expectation. Some gamblers have constructed a concept of expectation that is essentially equivalent to the mathematical concept.

GRAPHICAL REPRESENTATIONS: A STUDY ON THEIR PRODUCTION AND INTERPRETATION(*)(**)

A. Pesci, Dept. of Mathematics, University of Pavia, Italy.

The heuristic role played both in problem solving and in learning by different kind of graphical representations is well known. But we are convinced that their productive use is not always spontaneous: it needs practice and it requires giving suitable meanings to representations themselves.

With reference to a mathematical situation it is often essential both the activity of production of various representations and of translation from one type of representation to another.

The activity we have been carrying out with 11-12 year old pupils is mainly addressed to increasing their attention towards every graphical sign, so as to make them able to benefit from image code potentialities.

In order to reach such and aim we started with a didactic proposal which tended to foster suitable representations of given situations and to clarify the different peculiarities of various kinds of representations. This specific activity has been proposed for two years in 10 classes each year (about 400 pupils in all, aged 11-12) by respective teachers: each of them took part both in discussing the aims to reach and in formulating items for pupils.

The activity was composed of successive steps: a pre-test on basic skills of symbolisation (decoding and production), two working groups and one individual verification.

Discussion in class guided by the teacher following every step of group work was of fundamental relevance.

The discussion was centered on the production of the working groups and on the comparison with the others.

The main observations connected to representations produced by pupils to illustrate some situations (given in a verbal form) are the following (percentages refer to the last year but they are in accordance with those of the previous year):

a) A large part of groups (36%) cares only for qualitative differencies and doesn't care to specify the numerical link between the different symbols used.

b) A large part too (37%) uses drawings for particular cases: translating from verbal to figural form they lose every generality.

Another thing which may be stressed is that only 7% of the groups who uses typologies described in a) and b) employes 1 symbol for x objects (or persons). The majority employes 1 symbol for 1 object (or person): generalization is still far away.

c) The remaining groups (27%) use representations which correctly translate the general quantitative relations of the given texts.

All these and other observations were discussed with the pupils and were used to plan the successive didactic steps.

The last individual verification revealed that pupils had a good awarness of problems connected to the discussed graphical representations.

Teachers of our group agree on the necessity of spending a lot of time with the pupils to clarify the meanings of schemes and drawings in general.

Our opinion is that with this type of work pupils are forced to clarify for themselves the possible meanings connected to a graphical sign. In many situations they can therefore use schemes, diagrams, ... to translate situations in such a way to activate imagery and sometimes improve cognitive performance.

(*) Research supported by the C.N.R. and the M.U.R.S.T. (40%).

(**) The psychologist M.G. Grossi collaborated in the study.

STRUCTURE MAPPING WITH MANIPULATIVES FOR ALGEBRA

Cyril R.E.Quinlan
Australian Catholic University - N.S.W.

The role of manipulatives for introducing algebra to secondary school students is here examined in terms of Gentner's theory of Analogical models and Halford's Structure-mapping Theory.

Data were collected during 1990 from one school in tasmania and three schools in N.S.W.

The author observed, tested and interviewed students as they were introduced to algebra over a three weeks teaching period.

Follow-up, testing and interviewing were conducted six or seven months later.

Analyses of the results showed that the manipulative models used by experimental groups were generally useful.

Follow-up interviews indicated that students were not dependent on the models but were still able to benefit from their use.

Mappings from models to algebra epitomized the power algebra has to generalize several examples of a structure into precise algebraic form.

DEVELOPING MATHEMATICS READINESS IN YOUNG CHILDREN WITH THE AGAM PROGRAM

Micha Razel and Bat-Sheva Eylon

The Weismann Institute of Science

A group of preschoolers received two years of training in the Agam Program of visual education (Agam, 1981). Based on the assumption that school achievements in general and achievements in mathematics and geometry in particular are favorably affected by improved visual skills, it was hypothesised that training in the Agam Program would result in the child's developing a better general school readiness as well as a better readiness for school mathematics and geometry. Test results supported our experimental hypothesis (Eylon & Razel, 1986; Eylon, Rosenfeld, Razel, Ben-Zvi & Somech, 1990; Razel & Eylon, 1990, 1991). It was then hypothesised that these findings were due to skills of "learning to learn" in new visual tasks, that were developed in the trained group. Results from several test items, specifically designed to test learning in a multi-trial situation, corroborated this hypothesis.

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An Innovative Instructional Approach For Preservice Mathematics Teacher Training Influences Mathematics Instructors[1]

Vânia Santos
Indiana University and Universidade Federal do Rio de Janeiro

At present, society and media identify poor quality teaching as one of the primary causes of American students' low mathematics achievement in relation to other countries.. Our nation is demanding better mathematics teachers, yet too little effort has been made to understand what teachers or prospective teachers think about knowing, learning, and teaching mathematics. A noncognitive factor that current researchers of teaching and teacher education have not fully taken into account is teachers' beliefs about mathematics and mathematics pedagogy (Peterson, Fennema, Carpenter, & Loef, 1989).

Teaching actions are directly influenced by teachers' beliefs, and in turn those teacher actions have a tremendous impact on students' belief systems. Teachers, when they were students, often experienced mathematics classes that consisted of a predictable pattern of lectures followed by seat work. They are likely to teach in the same manner, perpetuating the long-standing chain of beliefs about mathematics as being: mechanical in nature, a fixed body of procedures that can be performed without thinking, activities that must be done independently, and, difficult except for those people who happen to be lucky enough to be good at it.

This study addresses the effects of an innovative mathematics content course (*Mathematics for Elementary Teachers Via Problem Solving* [T104]) on the instructors' mathematics belief systems. The subjects involved in the study were seven instructors of this new course for prospective elementary teachers. The primary question addressed was, How does the innovative instructional process used in T104 affect instructors' beliefs about doing, knowing, learning, and teaching mathematics? Data for the study were collected in several ways: classroom observations, teachers' interviews, and document analysis. Some examples of instructors comments include: "*I'm discovering that my students don't need me much anymore. They can handle problems on their own*" and, "*Sometimes I don't feel comfortable when students are unsure of what they are learning.*" Findings indicate that this innovative instructional approach has the potential to challenge teachers' conceptions and beliefs about learning and teaching mathematics.

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[1] This research was supported by National Science Foundation Grant No. NSFTEI8751478 to the Indiana University MEDC, John LeBlanc, Director. Opinions and conclusions are the sole responsibility of the author.

Mathematics Communication And Mathematics Teacher Education[1]

Vânia Santos and Anne Raymond[2]
Indiana University

Researchers interested in teaching and teacher education should explore alternative ways of preparing elementary teachers in an environment that in particular stimulates mathematical communication. It is important that teachers learn to clearly articulate mathematical ideas both verbally and in written forms, and the study depicted herein describes a mathematics course for prospective elementary teachers that attempts to incorporate an alternative pedagogy to the traditional lecture style (National Council of Teachers of Mathematics [NCTM], 1991).

The setting of this study is a new mathematics content course designed specifically for and required of prospective elementary teachers entitled *Mathematics for Elementary Teachers Via Problem Solving (T104).* In T104, students are actively engaged in group problem-solving activities where they are requested to share and negotiate meanings in their small group as well as articulate their ideas during the whole class discussions that follow. Assessment in T104 is an ongoing process where students receive continuous feedback via written and verbal communication with the instructor as well as through their participation in and reflections about special projects and presentations. Examples illustrating the key characteristics of this course will be provided during the poster session presentation. All the components of this course highlight the importance of clear and precise mathematical communication together with an additional emphasis on reflectiveness.

The primary focus of the inquiry was, How has the instructional process used in T104 stimulated students' abilities to communicate mathematics? The sample used in the study included eight undergraduate students enrolled in the new course during the first semester it was taught (Spring, 1990). Data for the study were collected in several ways: classroom observations, students' interviews, and document analysis. Findings indicate that the combination of problem solving and cooperative learning in an environment that encourages communication and reflectiveness has potential to improve students' abilities to articulate mathematical meanings.

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[1] This research was supported by National Science Foundation Grant No. NSFTEI8751478 to the Indiana University MEDC, John LeBlanc, Director. Opinions and conclusions are the sole responsibility of the authors.

[2] The authors are members of a team directed by Frank K. Lester, Jr. working on the T104 course described herein. Santos is also a mathematics professor at Universidade Federal do Rio de Janeiro.

AN EXEMPLIFYING SYSTEM FOR AN ARITHMETIC OF OPERATORS

Baruch Schwartz and Lauren Resnick
Learning Research and Development Center - University of Pittsburgh

Our general concern is with the growth of the concept of operator, i.e., with the structure of the four basic operations, and their main properties. This concept does not grow from interactions with the real world. For such a high-level entity, intermediate abstractions can build a bridge between symbolic manipulations and concrete situations. TRAINWORLD (Peled & Resnick, 1988) provides such a frame for an embodiment of operators. It consists of trains, machines and tracks corresponding to the different semantic meanings of additive and multiplicative word problems. **The Planner** is an enhanced version of TRAINWORLD which focuses on the manipulation of operators at a level of control. The children control the solutions of tasks by choosing a sequence of operators. The operator structures of problems are modeled, and the links between operators and semantic structures are made explicit. The Planner enables children to build, save, test and compare plans. Therefore the plan behaves as a "real" computer program. The growth of the concept of operator with the Planner depends heavily on the instruction chosen. One approach would use intermediate abstractions to model word problem situations, and then compare story problems between the different plans. The other approach would treat TRAINWORLD as a "world in itself" without reference to external situations. The student would acquire general schemes about operators and later match these schemes to real world situations. Two studies which reflect these two trends were carried out. Important issues about the role of intermediate abstractions in the acquisition of a concept, the role of situations vs a formal system in this acquisition, and about the promotion of metalevel skills, are raised by this research.

ON THE DEVELOPMENT OF THE MEASURE CONCEPT IN 13, 14, AND 15 YEARS OLD PUPILS

Dimitris Spanos, Tasos Patronis, Antonis Konstantopoulos
Department of Mathematics, University of Patras,Greece

In this report we want to communicate the first findings from a research project on the development of the measure concept in children in the age range 13-15. Eleven students (boys and girls) were called in long interviews to answer a question about filling the space of three non-structured shapes (rectange,trapezium and semicircle). Students' actions were clustered and identified as strategies and factors which influenced their geometrical intuitions were discussed.SRATEGIES a) Attempt to fill the space with squares or rectangles. b)Attempt tc fill the space with triangles. c) Attemt to fill the space with rhombuses. d) Filling the space with long thin strips:This strategy is of an intrinsic character and one can associate it with Cavalieri's heuristic method of the indivisibles. e) Filling the space with curved polygons: This is another interesting strategy introduced by some children.FACTORS INFLUENCING CHILDREN'S GEOMETRICAL INTUITIONS a) The gestal (from) of figure: This is a universal factor which seems to be present in each approach to all three figures. That is, when , for instance, the semicircle was given, the children immediately procceed in filling the space with curved stripes or with curved polygons and small circles homomorphic to the given figure.b)Social and cultural factors:The fact that students use the squares or the rectangles to fill a given shape can be viewed as a result of social and cultural habitual behaviour.c: The real context of the situation: The interviewer in order to simplify the mathematical model in social practice used a real situation of paving a room. Then the children's actions followed the tileman's system of filling the space which is a lived through experience by almost every Greek child as tilling is a wide spread social and cultural practice in Greece. It seems that tiles limit the students' intuition by imposing an "atomistic" attitude that works contrary to any infinitesimal processes that the students might tempt to involve

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Fifteenth Annual Conference of the International Group for the Psychology of Mathematics Educati
Assisi, June 29 - July 4, 1991

Proposal for Poster Session

Some Applications of Research on the van Hiele model of Thinking

Rosamond Tischler
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The van Hiele model of levels of thinking in geometry has motivated considerable research int children's learning. As an outgrowth of a clinical study (reported in Monograph Number 3 of the Journal for Research in Mathematics Education, The Van Hiele Model of Thinking in Geometry Among Adolescents. Fuys, Geddes and Tischler, NCTM 1989) the presenter has developed applications of this study in two areas – classroom teaching of elementary geometry, and the preparation and development of elementary and junior high school teachers.

Classroom materials to be presented were developed as an outgrowth of this study, and represent three types of applications of the research: (a) direct classroom implementation of approaches and materials used in clinical interviews in the above-mentioned study; (b) interpretations of the van Hiele levels with respect to geometry topics other than those in the study; and (c) a "levelist" approach to other topics in the mathematics curriculum.

One important objective in work with teachers (both preservice and inservice) is to communicate how geometry activities and teaching materials fit into the structure of the van Hiele levels. Other theorists have proposed other structures or sequences for organizing learning, for example by modes of representation, by levels of thinking, by degrees of abstraction, by methods of organizing classroom activities. A technique for developing teachers' awareness of such hierarchical structures is to have them classify teaching activities with respect to various systems. Such classifying activities encourage teachers to move to a higher level of thinking (in a van Hiele sense) about their own acts of teaching, for the objects of thought become not just the actions in the teaching activity, but the activities themselves. A similar movement to a higher level of thinking about teaching was described by Dina van Hiele-Geldof in her dissertation, "The Didactics of Geometry in the Lowest Class of Secondary School," (translated into English by the project that conducted the above-mentioned study). A format for developing and extending teachers' awareness of hierarchies in teaching activities will be presented.

A COMPARISON OF MATHEMATICS KNOWLEDGE OF OVERSEAS STUDENTS ENTERING A PRE-ACADEMIC PROGRAM

BARBARA ZINN AND YAEL VORONEL
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This study looks at the effects of gender and country of origin of students entering a pre-academic program for overseas students at the Hebrew University in Jerusalem. While students from all over the world are accepted, the two dominant groups at present are from South America (mostly from Argentina) and the USSR. Recent studies have shown the importance of students' prior knowledge as a dominant factor in their future academic achievements. All entering students took a placement test in mathematics.

This exam comprised 90 multiple choice items covering knowledge in six sections, namely: arithmetic, algebra, functions, trigonometry, exponents and logarithms. A sample of 713 placemennt tests of students from USSR and South America were examined using a two-factor ANOVA analysis.

Results showed that Russian students achieved significantly better than South American students in every subsection of the test. In addition Russian female students achieved better than their male counterparts on the whole test, while the opposite result appeared for South American students. A comparison of these two groups shows the effect of different systems of education on promoting achievement in mathematics and on encouraging women students.

addresses of the authors presenting research reports in the conference

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SOCIAL INTERACTION AND MATHEMATICAL KNOWLEDGE

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I shall address the problem of social interaction in mathematics teaching - learning process. Discourse analysis in traditional mathematics lessons shows patterns and routines that depend on implicit interaction rules and on an ideology of teaching as conveying ready-made knowledge. In recent years, some projects have been developed that aim to overcome the conveyance metaphor by means of an explicit alteration to the interaction rules of the classroom. Some trends in the related literature will be analyzed, as regards psychological frameworks, educational research models, criteria of choice for classroom activity, methodologies for data analysis. My thesis is that activity theory might offer a suitable perspective for research, complementary to constructivism, because of the major role of social dimension as a founding element of the theory. Finally I will present the research project on **Mathematical Discussion in Primary School**, *that aims at analyzing potentialities and limits of teacher orchestrated whole-class discussion at primary school level.*

1. Introduction.

In recent years the social dimension of mathematics learning has become manifest in the mathematics education research literature (e.g. Keitel & a.1989). We can consider the question at different levels (Margolinas 1989): (a) the **micro-didactics** (micro-level) that concerns the relationships between knowledge, teacher and learners (the **didactical system**) as regards a given item of knowledge and a given problem situation; (b) the **meso-didactics** (meso-level), that concerns the development of the states of the didactical system - as regards a given item of knowledge - when the problem situations change; (c) the **macro-didactics** (macro-level), that concerns the study of either the didactical system or the **education system** as a whole that unites the didactical systems within society (mathematicians, parents, politicians and so on) as regards their relationships with either knowledge as a whole or a part of it. For instance, at the macro-level, we have to consider that either the building of a body of knowledge (**knowledge to be used**) or its didactical transposition (**knowledge to be taught**) are social facts that depend on the cultural choices of society as a whole and transcend the individual's intention to teach (Chevallard 1985). At the meso-level, we have to consider the **planning of didactical sequences**, mainly as regards the choice of meaningful problem situations. There are some constraints, that are studied at macro-level, such as national programs; within these constraints different criteria of choice can be used that are pertinent to the cultural, social or cognitive plane: for instance epistemological analysis (Brousseau 1986), to focus on the historical genesis of mathematical concepts, or analysis of everyday mathematics (e.g. Mellin-Olsen 1987, Bishop 1988), or analysis of pupils' conceptions (e.g. Steffe & a. 1988). At the micro-level, we have to consider the involvement of all the participants in the **social interaction** process that takes place in every classroom activity. This aspect is an element of Brousseau's (1986) **theory of situations** and related research on the analysis of situations (see Laborde 1988 for a review). As a result, some theoretical constructs are at our disposal to analyze classroom events: I quote only **the didactical contract** (Brousseau 1986), that is *a set of rules which organise the relationships between the content to be taught, the pupils and the teacher within the mathematics classroom.*

The analysis of didactical phenomena has usually to be carried on at a specific level to avoid ambiguities. But the levels are strictly interrelated. For instance, the education system (macro-level) is both an influencing environment for all the research projects that are developed at the meso- and micro-level and an organism that develops together with its components. Micro- and meso-levels of analysis are related too: for instance the meso-level concept of classroom **custom** (coutume - Balacheff 1988), that makes explicit *the way how the social group* (i.e. the classroom) *interprets the social relationship,* has to be considered in the analysis of the negotiation of a didactical contract.

The interest of PME community in the social construction of knowledge is growing. The plenary lecture given by Bishop (1985) was followed by the creation of the working group *Social Psychology of Mathematics Education*. Some contributions were presented in recent years at PME conferences (for a review see Balacheff 1990a, Laborde 1990), but the research field is far from a systematization, because of the lack of worldwide, accepted theoretical and methodological tools. Nevertheless, some trends in the research literature are recognizable, as regards at least the subfield of **social interaction**, as I shall argue in the following section. I shall focus mainly on **verbal interaction**: surely verbal interaction is not the only instance of social interaction in the classroom, but it is probably the most important, because the institutional goal of schooling is the transmission of accumulated knowledge and this transmission takes place mainly by means of language (Pimm 1987). Verbal interaction in traditional classrooms is regulated by very strong social rules which seem to be followed though the actors are not aware of them (Edwards and Mercer 1987). The rules are usually very different from everyday talk rules (e.g. teacher can ask questions even if he already knows all the answers) but are implicitly accepted. In traditional mathematics lessons it is possible to identify complex routines (Bauersfeld 1990) that are not so much related to mathematics as to social conventions and have the very effect of maintaining the perceived status quo of school mathematics. Traditional classrooms are influenced by a teaching ideology that consider teachers as conveyors of ready-made knowledge. The **learner's responsibility for learning** has been stressed, even if with different emphases, by several approaches to learning that contrast with behaviourism. So, in recent years several research projects have been developed that share the common goal of overcoming the conveyance metaphor by means of an explicit alteration to the verbal interaction rules in the classroom. In the following, I shall discuss some exemplary projects that may be classified as **second generation studies**, as the focus is not only on the effects but also on the process of social interaction. The projects are supposed to refer to the same **philosophy of mathematics** (the so-called **social constructivism** - Ernest 1991 - that views mathematics as a social construction) but refer to neither the same psycological framework nor the same pedagogical model and clearly show that there is no one-to-one correspondence between background theories and forms of pedagogical practice (Kilpatrick 1987).

2. Some Trends in the Mathematics Education Research Literature.

2.1. Psychological frameworks for research on social interaction.

The problem of social interaction is considered differently in the foundations of general approaches to learning. For instance, social interaction processes have not played a popular role in **cognitive science**, at least as regards the strong program of constructing explicit, implementable models of complex psycological processes (Newmann & a. 1989 for a discussion).

On the contrary, **activity theory** (Vygotsky, Luria, Leont'ev, Davydov) and **radical constructivism** (von Glasersfeld 1990) have explicitly taken into account the problem of relationships between social interaction and learning. Activity theory and radical constructivism have many common ancestors and many related similarities, but their perspectives are different, (for comparisons, see for instance Bauersfeld 1990, Raeithel 1990). Constructivism is very popular in PME community, as the focus on the learning subject, that is a fundamental element of radical constructivism, fits well with psychological analysis of the process of conceptualizing reality of an individual child (e.g. Steffe & a. 1988, Vergnaud 1991). In a constructivist perspective, *(social interaction is the process) by which people interpret each other's actions and thus achieve compatible meanings* (Cobb & a. 1990).

The so-called **European approach to social cognition** or **social constructionism** assumes a different position and tries for a coordination of Piaget's original theory with Vygotsky's contributions. The focus is well expressed by the famous Vygotsky's quotation (1978): *Any function in the child's cultural development appears twice or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category.* The process of transformation is called **internalization**. As experimental research has proved that the advantage of joint activity cannot be reduced to imitation, the **socio-cognitive conflict** (that is a *conflict, socially experienced, in which an individual 's strategy is explicitly contradicted by another person's strategy* - Perret Clermont 1980) is proposed as a possible mechanism for internalization by social constructionism, that analyzes too the conditions for its functioning.

This point allows to distinguish between possible psychological frameworks of research on social interaction that induce different developments of process analysis. Social constructionism is supposed to frame some European research projects, such as **Diagnostic Teaching** (Bell 1984) and French research based on the theory of **didactical situations** (Laborde 1988), while (radical) constructivism frames Bauersfeld's **interactional studies** (1990) and some American research projects such as the **Purdue Problem-Centered Mathematics Project** (Cobb & a. 1990). These distinctions are not to be taken in a rigid way: for instance, in the Purdue Project (Cobb & a. in press) the initial constructivist perspective has been coordinated with contributions from Leont'ev's papers, in order to analyze the relationships between the classroom and the wider socio-political setting. Coordination between different theoretical frameworks might be considered as an example of **complementarity**, as it is described in Steiner's proposal for TME (1984): the principle of complementarity requires simultaneous use of descriptive models that are theoretically incompatible.

Activity theory is not very popular in PME community, in spite of the fact that social dimension of learning is a founding element of the theory. Some key concepts of activity theory might be very useful in framing research on social interaction in mathematics education. Consider the central role attributed to the mediation of tools, in particular signs (e.g. speech), in what is called **semiotic mediation** (Vygotsky 1978) for the development of every human activity and the role of teaching in cognitive development. The teacher is the responsible not only for organizing a suitable and encouraging environment for learners and for provoking conflicts but also for providing direct guidance in solving problems and in constructing meaning. The last observation explicitly refers to Vygotsky's famous definition of the **zone of proximal development** (1978) as *the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult's guidance or in collaboration with more capable peers.* The analysis of the process of internalization of collective activity and of the conditions of its functioning within the zone are still open problems in activity theory (Davydov 1990): mathematics classrooms are suitable settings for further research (§ 2.5).

2.2. Theoretical models of educational research.

Research on mathematics education may be influenced by models of general educational research. In the following I shall briefly describe two models.

Most of the French researchers have chosen an approach that aims at *describing the functioning of teaching situations as the functioning of a system that depends on choices and that is submitted to constraints, at loosening the constraints and choices and at finding how different choices cause different learnings* (Laborde 1989). This model borrows some methodologies from scientific research in experimental science, in the sense that it allows (at least theoretically) the analysis of situations with systematic changes of variables. So, it is possible for instance to approach the problem of **reproducibility**, with a quantitative model that allows, by means of a computer simulation, the testing of *the respective weight of variables* and the analysis of *the influence of their variations on the system dynamics* (Artigue 1986).

Other research projects are based on a different model: finding or provoking by trial and error effective teaching situations and looking for a description of the variables that assure "good" learning (e.g. the Purdue project, Diagnostic Teaching). The experimental work is often carried on as **action-research** in cooperation with teachers while retrospective analyses are presented as **case studies** with controlled extension to a small group of classrooms: reproducibility is problematic because for instance the experiments could rest on the exceptional features of experienced teachers who have expert knowledge.

The difference between the models is evident if we compare the **teacher's role** in the research development. In the former approach, the teacher is supposed to be an **object of observation**: this fact eliminates a lot of methodological problems, and allows the production of materials that may function because of the intrinsic features of the situation, but might run into the problem of **teacher's responsibility for teaching** (that is the counterpart of learner's responsibility for learning in the classroom "game"). In the latter approach, the teacher is either **actor** or **observer** and runs into a paradox (Brousseau 1986): he cannot be an actor and an observer of his own actions in the same time, but he must be an actor and an observer in order to make decisions in the course of interaction. Ethnographic methodology of **participant observation** (Eisenhart 1988, Davis 1990) is necessary.

The model of action-research is more coherent with the assumptions of both radical constructivism and activity theory. According to Goldin (1990), *[if we accept radical constructivism], descriptive case study must replace controlled experimentation in the assessment of mathematical learning and teaching effectiveness, because the cognitions of individuals are simply not comparable.* In activity theory, the necessity of overcoming the model of natural science is explicitly stated by Raeithel (1990) in his theoretical approach to modes of reflection: ***1. The position of the naive problem solver: Primary entering or sorting out symbolized possibilities*** *[...] The first mode of such a reflection is entered when the flow of action is broken by events that were not anticipated, and the subject must turn into an observer [..] The subject remains naively centered in herself [...] the meaning of objects is still inherent to them, i.e. the symbolic structure is inseparable from the perceived reality [...]* ***2. The position of the detached observer: De-centered analysis of the functionality of means.*** *A second, de-centered mode of reflection is entered when the subject observes another subject's activity [..]. The subject can even try to see herself from a distance [...] and this shows that the full power of de-centered reflection may only be reached with highly developed symbolic means, because in the case of really tough and urgent problem the flow of activity is so complex that it is to be re-presented in a model of the process. [..] The ability of humans to invent natural sciences is seen as a generalization of our ability to observe and understand other humans [...]* ***3. The position of the participant observer: Re-centering, or producing the voice of a community.*** *[...] Another, still higher mode of reflection is necessary to re-establish the freedom and power of human reality production: we may* ***choose*** *(in the fullest sense of the word) from the possibilities, to find the ones that* ***should*** *be turned into reality, if feasible.[...] This mode of reflection is once again centered, in its highest form, not like the strategic action of a single economic actor, in an individual subject, but in the community of which the subject is a member. This also means that the split between observing and observed subject that formed the starting point of the decentered mode will now be developed into a dialogical relation [...] .*

2.3. The context and situation of classroom activity.

In his lecture at PME, Bishop (1985) asserted that *if there is one thing to be learned from research into social aspects of mathematics education, it is that the context and the situation are all important.* The most interesting challenge is, in my opinion, to consider the context and the situation in all their educational aspects. Consider, for instance, the classroom experiences on voting bodies (described by Steiner (1990)) which *provide us with a learning context for students in grades 11-12 and an educational setting in which social cognition and social learning, metacognition, learning about learning, communication about communication play a significant role and can be made a matter of in depth didactical investigations. It is considered important that, in the context and setting specified, these factors are not seen independent from the content and its epistemological structures. Rather, they are viewed as being profoundly connected with the constitutive interrelation between theoretical concepts, applications, knowledge development and social interactions in the broad field of mathematics related activities in science, education and practice, understood as a socio-historical reality.* Therefore, in Steiner's project, classroom organization in small-group work or discussion is not an a priori choice but strictly functional to the selected context.

But the choice and didactical use of suitable contexts is a hard problem in didactics of mathematics. As Boero (1989) asserted in his lecture at PME, usually *the problem of the choice of context is taken into account only from the point of view of the specific mathematical learning aims, with a resulting "episodic" feature of the "contexts" in front of the organic unity of mathematical curriculum.* So he was lead to integrate the existing theoretical tools with the concepts of **field of experience** *(a sector of the experience (actual or potential) of the pupils identifiable by them, with specific characteristics which render it suitable (under the guidance of the teacher) for mathematical modelling activities, posing of mathematical problems and so on)* and of **semantic field** *(an aspect of human experience (real or potential) which appears to the researchers, in one or more fields of experience, as unitary, which cannot be broken down further, and which can be rationalized through pertinent, intense and meaningful use of mathematical concepts and/or procedures).*

Context is supposed to play a major role in shaping individual learning processes and cognitive development (Boero 1989). From a social perspective, a context that is well known by only a minority of pupils makes it difficult to gain participation from the whole group; a context that hints at questions makes it possible to solicit answers, to look for comparisons and contrasts by means of individual arguments. So we can justify the hypothesis that *context affects also social interaction processes and their effects on individual cognitive development.* A detailed analysis of this relationship is an interesting problem for future research. School is not the only setting where it is meaningful to study mathematics learning: there are **non-formal settings** too, such as supermarkets, family, streets (e.g. Carraher 1988), where context, implicit interaction rules and the use of semiotic mediation tools are different. Comparison between different settings as regards social interaction processes might contribute to the study of problems of internalization.

In some projects (e.g. Steiner 1990) **the context itself determines the choice of learning situations.** This approach fits well with the perspective of activity theory, that focuses on **object-oriented activity** rather than on knowledge (Leont'ev 1975-77). In other projects the context is not explicitly chosen among possible ones, so that classroom activities are determined on the basis of other criteria. For instance, in the Purdue project external constraints are supposed to play a major role on focusing on arithmetic as the main subject area for 2nd grade mathematics. Then, the choice of problem situations is made on the basis of an explicitly-declared criterion: the cognitive models developed by Steffe & a. (1988) for the construction of aritmetic meaning. In Diagnostic Teaching, the problems have been chosen on the basis of a previous analysis of pupils' misconceptions (Hart & a. 1981). Surely we have no universal criterion, but what is important in order to look for possible comparisons of results and methodologies is that **the criterion is explicitly declared.**

In this section I have explored some problem at the macro- and meso-level. If we shift to the micro-level we may question which features of the task can affect the functioning of a situation of social interaction. Micro-level analysis is the core of French research on the theory of situations (Brousseau 1986), that distinguishes several types of situations. We are especially interested in **situations for formulation** (where the pupils have to make knowledge function in order to produce formulation; they are called **situations for communication** when they have explicit social dimensions) and in **situations for validation** (where the pupils have to produce proofs and thus make explicit the related theories and means underlying proving processes). A review of French research focused on social dimensions of mathematics teaching and learning is in Laborde (1988). In this lecture, I shall quote from the paper only some limits in the functioning of such situations, that have to be taken into account in the planning of classroom activity. When pupils are set a task to be solved together, if the **cognitive distance** between the members of the group is too large, it is not possible to ensure the understanding of other's arguments; if the **time** allotted is not sufficient, it is not possible to develop effective interaction; if the task allows a solution based on individual perception (such as some geometrical problems) it is not easy to induce pupils to distance themselves from action and to produce a rational argument.

2.4. Analysis of Classroom Data.

The analysis of classroom data is based on an elicitation of the facts that are pertinent to the interpretation of didactical phenomena. The data of research on social interaction in the classroom usually consists in videotaping or audiotaping the lessons, in written transcripts of verbal interaction, contextual information, interviews with pupils and teachers, and so on. It is not possible to have a general agreement on the methodology of data analysis because it depends on the way of looking at social interaction processes, that depends on one's theoretical framework. Nevertheless, the results of the previous paragraphs, in spite of their fragmentation, suggest a very interesting research problem: the individualization of different processes of interaction and their description as regards the roles of the participants and the relationship with individual processes. In some projects there are attempts at distinguishing several types of

classroom discussion: these distinctions have still a "local" character, because they are usually strictly related to the particular project. I quote just some examples.

Legrand (1988), in his study of **co-didactic situations**, used the scientific debate at University level to introduce proofs as functional tools. He distinguished several kinds of debate: **modelization debate**, that concerns the introduction of a new notion, by means of a problem, proposed by the teacher; **adequacy debate**, that takes place when a notion or a concept has already been studied, asking students to furnish problems whose resolutions require the use of new tool; and **conjecture debate**, that concerns the discussioon of some conjectural statements, proposed by the teacher or by the students themselves.

Interactions between teacher, pupils and knowledge are considered in research based on Brousseau's theory of situations in different process phases (Margolinas 1989). Among them there is the so-called **balance** (Douady 1984), that follows a phase of individual or small-group work and aims at fostering pupils' formulation of their strategies, at disseminating methods and giving the teacher the opportunity to take part in the debate when the outcomes do not fit his expectations. In this case, the teacher's role is not strictly determined in advance and has to run into his personal beliefs (Margolinas 1989). Perrin-Glorian (1990) has studied another type of situation : the **recall,** where pupils are called to account orally for a situation of action which has taken place in a previous session, when action is no longer possible. The recall is supposed to be useful to low attainers in mathematics classrooms.

In the Purdue project there are distinguished two major levels of discourse with different patterns of interaction: **talking about and doing mathematics** and **talking about talking about mathematics**. At the first level, the teacher aims at involving the students in the process of negotiating mathematical meaning . At the second level, the teacher chooses as explicit topic of conversation the social norms of interaction in the classroom and is much more directive.

Consider now the problem of **analyzing the written transcript** of a small-group interaction or of a classroom discussion. The standard schemes of talk analysis (Edwards & Mercer 1987 for a review) do not seem very useful, because they deal with the form of what is said rather than with its content. In the literature on mathematics education two different approaches to transcript analysis have been proposed by Voigt (1985) and Steinbring (1988) (Seeger 1988 for a comparison). Voigt's analysis takes the perspective of social interaction and discovers **patterns** and **routines** in classroom talk, while Steinbring's analysis starts from a conceptual model of mathematical meaning that distinguishes three interactive levels (**sign, object** and **concept**). Voigt's analysis is focused only on the sociological facet of the process and leaves in obscurity the specificity of mathematical knowledge, while Steinbring's analysis is focused on an epistemological model of mathematical knowledge and does not take into account the richness of the contribution of all the participants in the interaction: they seem complementary (§ 2.1), because it is necessary to take into account sociological as well as cognitive perspectives. A tentative solution, that is adopted in several research projects is presenting substantial excerpts of situated classroom talk, transcription of more or less raw speech with information about contextual activity, in order to describe the exchanges between teacher and pupils taking into account either their subjective interpretations or the direction of the talk towards the construction of mathematical meaning. This kind of reporting is similar to a diary, a ship's log or an expedition report (Engstrom 1987); even if the analysis is obviously made up later, it strives for taking into account the instant subjective reconstruction of meaning and for limiting outsider's wisdom, easy to profess after the events are over.

2.5. The problem of internalization.

Micro-level analysis of social interaction processes in particular tasks leads to discussion of the problem of *internalization of social practice as a higher order function* (Vygotsky 1978). This problem is not pertinent to a radical constructivistic perspective, because *social interaction is not a source of processes to be internalized* (Cobb & a. 1990). But the problem is crucial to a vygotskian perspective (§2.1) and requires a content-bound analysis because of the **specificity of mathematical knowledge.**

Consider the problem of approaching **mathematical proofs.** Balacheff (1990) has studied the students' treatment of a refutation, by means of social interactions which encouraged the confrontation of different viewpoints His work confirmed the benefits of social interaction but enlightened its limits too, because of the major role played by argumentative behaviour: *they* [argumentative behaviour] *are genuine epistemological obstacles to the learning of mathematics [...]; argumentation and mathematical explanation are not of the same nature: the aim of argumentation is to obtain the agreement of the partner in the interaction, but not in the first place to establish the truth of some statement [...] Mathematical proof should be learned "against" argumentation, bringing students to the awareness of its specificity and of its efficiency to solve the kind of problem we have to solve in mathematics.* In my opinion, mathematical proofs too might be internalized from social interaction practice, on condition that the dialogue involved not only the partners in the classroom but also the representatives of mathematical culture (i.e. the mathematicians), that give the rules of acceptance of a proof at a given time and at a given level of rigour. This dialogue could require *semiotic means other than speech* (Raeithel 1990), such as written texts. However, the negotiation of the level of rigour is a matter of didactical contract and the teacher's role is, in this case, irreplaceable.

Another example of internalization of social practice is offered by the ongoing research on **hypothetical reasoning in problem solving** (Boero & Ferrero 1991). Making an hypothesis is considered as a *speech act that communicates, after a suitable question, the image of a possible reality, selected, as for the questioner, from a range of possible answers, and, at the same time, as thought act that underlies the imagination of possible reality.* As a result, the social practice of introducing, contrasting and verifying hypotheses is a source for the individual process of hypothesizing. A further example will be discussed in the following (§ 3.4).

3. The Mathematical Discussion in Primary School Project.

In this section, I shall present some general features of the *Mathematical Discussion in Primary School Project* that is in progress in my research group, as a cooperation between a group of primary school teachers and me. It is a part of a more general project on **innovation in mathematics teaching** that shares with other Italian projects some interesting features which can be briefly sketched as follows: it is an **action research project,** where the research and executive staffs are the same and the teachers are in the classroom as **participant observers;** it is focused on **long run processes** and **general educational goals** as the teachers teach (not only mathematics) the same group of pupils for five years; it considers advanced thinking problems and the **construction of complex reasoning,** even at primary school level; it aims at a program of **mathemathical literacy for all** (Boero 1989); it introduces a **cultural and historical dimension** in mathematics teaching. The history of mathematics may be a source for didactics from several perspectives. On the one hand, it contributes to the didactical analysis of a particular concept (e.g. numbers; arithmetic operations; volume) This analysis is a fundamental element for the *planning of classroom materials,* that might refer implicitly (e.g. choice of situated problems to be posed to pupils) or explicitly (e.g. excerpts from original papers, to be read in the classroom) to the historical development. Both uses of history are important: the former may give the pupils the occasion to reflect on the cultural development, whose mathematics too is a part, and foster *the awareness of being a member of a social group larger than the classroom or family;* the latter may give the pupils the occasion to elicit, to confer dignity on and to overcome some epistemological obstacles that are necessary steps for building knowledge: so historical development gives *a part of the meaning* of a mathematical concept. On the other hand, in the *analysis of classroom data,* history is very useful for *the study of pupils' conceptions* and their development.

The research arose in 1986 from a didactical problem: every teacher needs to manage **whole-class activities**, as interaction with the whole group of pupils. Educational research usually takes care of context-free situations where the specificity of mathematical knowledge is lost; on the contrary, we were especially interested in **mathematical discussion** (Pirie & Schwarzenberger 1988), that is *purposeful talk on a mathematical subject in which there are genuine pupil contributions and interaction.* The main goal of the project is **checking potentialities and limits of teacher-orchestrated mathematical discussion at primary school level**. The development of **strategies for managing discussions** has been coming out as a relevant by-product of the research project.

The project was developed in three phases: at first the whole group tried for the building of a tentative **theoretical framework,** that might have given us a rationale to focus on discussion; then the group tried for fitting discussion into mathematics classroom in spite of the major problems of contrasting school custom; and finally the group began the **second generation study** in oder to analyze the processes of interaction (this phase is still in progress). The early outcomes of the project consist of (a) building of theoretical framework (§ 3.1); (b) determining conditions to begin the classroom experiments (as regards classroom custom and teacher's role (Bartolini Bussi 1990)); (c) determining the objects of classroom activity (§ 3.2) and (d) the main modes of discussion (§ 3.3); (e) carrying on classroom experiments; (f) analyzing classroom data.

3.1. A Sketch of the Theoretical Framework.

At the very beginning of the work, we referred to **social constructionism,** yet we were especially interested in those kinds of interactions where an adult is present (the teacher) who aims at fostering mathematical learning, while most laboratory experiments focused on **peer interaction**. In other words we were interested in what happens in the **zone of proximal development** (§ 2.1). The influence of Vygotsky's and Luria's papers was very deep from the very beginning, but we do not consider ourselves as scholars of activity theory, even if as work goes. on key concepts of activity theory seem to fit better with our research project. The focus on the teacher's role in guiding pupils' work and on the individual internalization of collective activity has detemined the development of methodologies of transcript analysis.

From the literature on mathematics education, we made the hypotheses that discussion might have been a suitable setting *to overcome the contextuality and subjectivity of mathematical experience; to elicit the systemic structure of mathematical ideas,* that is hardly evidenced by individual solution of particular problem; *to introduce and to explore the functionality of psychological tools (at first speech) in mathematical learning; to nurture metacognition; to* *modify (in pupils and teachers) some stereotyped beliefs about the nature of mathematics*. The early case studies validated the hypotheses (Bartolini Bussi 1989).

The methodology for the experimental part included (Bartolini Bussi 1989): **seminar activities** (i.e. activity carried on together by the whole group of researchers): a priori analysis, outline of classroom activities, retrospective analysis, evaluation and assessment; **individual activities** (i.e. activities carried on by each teacher with only support from the seminar): detailed planning of classroom experiments, organic integration of the activity in everyday school life, classroom observation, as regards recording and transcribing classroom discussions, collecting individual protocols, giving contextual information to frame the situation.

3.2. The Objects of Classroom Activity.

At the very beginning, we had a naive attitude towards the **objects of classroom activity** and believed that, in principle, every item of knowledge might have chosen as a focus to plan classroom experiments, providing suitable problem situations. In the early experiments, the problem of **context** (§ 2.3) was not explicitly considered. But, in the course of the work, as a result of our retrospective reflection and of interaction with other Italian research groups(especially the Piossasco group - Boero 1989), we became aware that our choices had been implicitly influenced by some beliefs. **The object of classroom activity has to be chosen because of its relevance to (a) the development of mathematical knowledge,(b) the relationships with historical development of mathematics, (c) the potentiality of expending outside school, (d) the adequacy to pupils' cognitive development.** In other words, we tried for examples of development of the **cultural approach to the mathematics curriculum** (Bishop 1988).

The very early partial experiments were carried on in the early grades of primary school and even at pre-primary level, as regards the early approach to natural numbers. As soon as pupils has experienced some counting activities (inside or outside school) to solve relevant problems, we began to put to them a very provocative question: *what are numbers?* It is an early example of what we call **conceptualization discussion**, that will be more precisely described in the following. As the provocation seemed to work well even with very young pupils (aged four of five), we began the controlled experiments and carried on four case studies focusing on: **infinity** (4th grade), **volume** (5th grade)(Bartolini Bussi 1989), **graphs in the cartesian plane** (5th grade) (Bartolini Bussi 1990) and **coordination of spatial perspectives** (1st and 2nd grades) (§ 3.4).

3.3. Modes of Classroom Discussion.

We have distinguished three main modes of mathematical discussions, relevant to our project. They may be described as follows.

The **mathematization discussion** (or problem discussion) raised by a text or an open problem situation that may be modelled by means of mathematical concepts or procedures. It can be a **problem-solving discussion** (that is a collective solution of a problem) or a **balance discussion** (that is socialization and collective evaluation of strategies that have been set up by the pupils in individual or small group work). The latter is similar to *whole-class discussion* in the Purdue Project and to the *balance phase* (Douady 1984) but has some particular distinguishing features. It usually takes place some days (even one or two weeks) after the individual work: the delay fosters the **distancing** of the pupils from their own product so that it may be considered in a detached way; moreover, the teacher does not propose the comparison of all the strategies but selects only the prototypes of different strategies or maybe different problem representations, then he invites pupils to say which is the strategy that is more similar to their own. Then everybody has to take part in the discussion to defend (if it is still necessary) his own position. The aim is to agree on one (or more) consensual strategies, but if it is not possible to agree on a strategy that is coherent with accepted mathematical behaviour, the teacher istitutionalizes the fact that it is not possible to have a shared strategy and that the problem has to be considered again after some time: the acceptance of periods of uncertainty is a relevant feature of classroom custom (Bartolini Bussi 1990).

The **conceptualization discussion** (or weaving discussion) is raised by a provocative question as *What are numbers?* or *What is a graph?* or *What is infinity?* or *What are negative numbers?* It is used to induce pupils **to construct or to enrich the meaning** of some experiences (that have taken place inside or outside school), where mathematical concepts might have been used only as tools to solve problems. The task is possible with very young pupils too because of the very existence of a particular **word** (e.g.number, graph, infinity) and of a related lexicon, in mathematics and in everyday language. Here it is evident the importance of focusing on themes that had as great relevance in the cultural development as to have influenced the development of everyday language. In such discussions pupils cope with a problem that recurs in the history of mathematics: **defining a concept** in order to put it in the body of scientific knowledge, just when it has been only used as a tool to solve particular problems. So pupils are faced at true **epistemological obstacles** and their formulation are often very similar to the formulations of some mathematicians of the past and may be compared with them. Conceptualization discussions may be proposed (a) at the very beginning of a teaching sequence to collect and share individual experiences and (b) after some school activities (e.g. individual tasks or balance discussions) to foster the collective weaving of a conceptual network that contains the given concept. This discussion gives the teacher the occasion to observe the development of initial conceptions and to adjust the planning of other activities towards formulations that are more and more similar to the socially-recognized ones and that elicit similarity and contrasts between the mathematical and everyday **registers** (Pimm 1987).

Mathematization and conceptualization discussion might be related to **the dialectic between tool and object**, that is analyzed by Douady (1984). In mathematization discussion, the tool aspect is more important, while the conceptualization discussion allows to consider concepts as *cultural objects that have a place in the body of scientific knowledge, at a given time, and which are socially recognized.* This dialectic between tool and object is not only a feature of the whole sequence but is an element of every conceptualization discussion, because of the continuous shift from the plane of concrete (or mental) actions to the plane of mathematical concepts and vice-versa. It is consistent too with Davydov's description of **complementary aspects of empirical and theoretical knowledge**: the former *is produced by comparing objects and their representations which makes it possible to discern in them common general traits;* the latter *arises on the basis of an analysis of the role and function of a certain relation of things within a structured system* (Davydov 1972-1979).

The **meta-discussion** is raised by the proposal of negotiating interaction norms, of evaluating the task or the discussion itself as a teaching aid, of evaluating the whole didactical sequence (the term meta-discussion is justified by the fact that there is evidence of a lot of meta-cognitive activity). It may be introduced directly by the teacher or indirectly by reading together the transcripts of a previous discussion. In the latter case, it aims explicitly at the construction of the **history of the group**. It is the suitable occasion to *talk about talking about* *mathematics*, without authoritative and evaluative utterances of the teacher.

A **typical classroom experiment** consists of individual task and discussions that are intertwined in a functional way. The same discussion may consist of different phases (mathematization, conceptualization and so on). Up to now, the shift from one phase to another is an outcome of expert teacher's knowledge, but, as the experiments are carried on in at least three classrooms in the same time, we are looking for making structural regularities explicit that might give hints for replication.

3.4. Some remarks from a case study.

Now I shall refer to the case study of **coordination of spatial perspectives**, that was worked out as a classroom experiment in the school year 1988/89 (1st grade) and 1989/90 (2nd grade) in three different classrooms of different schools (Boni & a. to appear).

The **objects** of classroom activities may be described as regards **motives** (Leont'ev 1975-1977) for the participants in classroom interaction. The motive for the pupils was *representing things by means of some system of signs* (gesture, speech, drawing, written language). The motive for the teachers was more complex and twofold as it included: **(a)** *fostering the coordination of spatial perspectives*, as the representation of the global image of either an object or a set of objects by means of the organization of partial images (perspectives) obtained from different points of view (Samurcay 1984); **(b)** *fostering the development of different modes of reflection*, from primal centering to decentering to recentering (§ 2.2; Raeithel 1990). So the pupils' motives were concerned with the object as a **field of experience**, while the teachers' motives were related to the object as a **semantic field** (§ 2.3).

The choice of the object is consistent with the criteria discussed above: (a) it is relevant to mathematical knowledge, because it allows and fosters reflections on either three-dimensional geometry (the world) or two-dimensional geometry (the drawings), on their similarities and contrasts and on their representation by means of other systems of signs (such as speech); (b) the socially-accepted results (that is, the development of drawing systems, the theory of linear perspective and finally the descriptive and projective geometry) are the outcomes of a very complex process that produced some of the most visible products of our culture (Panofsky 1924-1984); (c) the expansion outside school is "natural" (e.g. work of art analysis; photography); it fits with 1st and 2nd graders cognitive development, as it is proved by the very fact that most of psycological research on spatial representation deals with children at the beginning of primary school; besides, the national programs for Italy claim the development of early spatial frames of reference in the first two years of primary school.

The tasks of the experiments (**actions** according to Leont'ev 1975-77), were mainly based on individual coding (and decoding) of some part of the visible world by means of systems of signs (gesture, speech, drawing, written language) and on collective comparison of products and strategies. **Gesture** was considered as both **externally oriented** (e.g. handling objects) and **internally oriented** (e.g. touching or looking at parts of an object as an early operation before drawing it) (Vygotsky 1978). **Speech** too was considered as both externally oriented (e.g. discussing with others) and internally oriented (e.g. **inner speech** in individual problem solving). **Drawing** was not considered as an evidence of pupils' cognitive development but as a **problem solving task** (Freeman 1980), where *a lot of different graphic decisions had to be made at various stages in the process of production of a figure* and where the recourse to stereotypes has to be considered as a powerful algorithm in solving complex problems of representation. **Written language** was introduced as soon as possible either in individual problem solving (to **allow** later analysis) and in institutionalization phases (to build the history of the group). As the term **point of view** had been spontaneously introduced by some pupils, at the end of the 1st grade, we proposed a conceptualization discussion: **what does point of view mean?** It was a great move towards theoretical thinking because 1st graders were induced to construct and to enrich the meaning of some fragmented experiences. In the second year, real-life drawing was the main goal of the experiment together with collective discussions of the products and of the strategies. Drawing objects (cubes, tables) with a very simple geometric structure made it possible for the teacher to induce the necessary conflict between the geometrical properties of three-dimensional shapes and those of their two-dimensional representations.

It is not possible to describe the whole sequence; so, I focus only on some particular tasks: ***(a)*** *a balance discussion (1st grade);* ***(b)*** *a conceptualization discussion (1st grade); (c) a balance discussion (2nd grade).*

(a) *It took place just after some months of school. Two pupils had taken part in a communication game, where the task of the coder was the following:* You have to build a scene with these toys (a large house, a horse, a car, a puppet and some trees) and while you are building it, you have to describe what you are building to your friend that is behind that screen. He will build the scene. Then we'll take away the screen and compare them. *The whole classroom had taken part in the comparison. Then each pupil had to draw the scene from his own point of view, exactly as he saw it. The drawing had to be compared (balance discussion) to evaluate collectively their correctness. This sequence - coding/decoding, discussion, drawing, balance discussion) was repeated twice at the beginning of 1st grade with two different pairs of coder-decoder. I now refer to the last balance discussion. The teacher invited pupils to evaluate all the drawings, with respect to the particular position of the drawer. Then the teacher invited the pupils to explain their difficulties and to evaluate the "new" learning. The following excerpt (emphasis added) refers to the latter aspect (meta-discussion phase).*

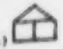
(1) **Elisa:** I've learnt that one can put the things...when we were in the kindergarten, we put all the things in front. Now I've learnt that the things may be put behind and that one can joke with things.
(2) **Ingrid:** I made the house all in front, but now I've understood that there are curves too.
(3) **Francesca:** and that one cannot see the whole **thing**, but maybe only some pieces, that one has to draw the **fronts** you see and not only what you like.
(4) **Stefano:** I've learnt that if you say to draw a **thing** I cannot do as I like but as **they are there**, as the teacher puts them.
(5) **Greta:** We in the kindergarten put the things as we liked, where **they were held**..now I've understood that we have to try and put them in their place. ......
(6) **Teacher:** [..]Giusi and you..have you learnt? Do you remember what you did? A thing that you do not do any more.
(7) **Elena:** She drew very small...
(8) **Giusi:** I drew out.
(9) **Teacher:** And now...
(10) **Giusi:** I make the desk large, then the house and then the others.
(11) **Francesca:** To better understand, I put the sheet so *(and she lifts the sheet vertically).*

In this very brief excerpt we have pupils of different level. Giusi (8-10) in the first drawing had not succeeded in putting all the objects on the table, because she (maybe) had not been able to organize the space. She was induced by the teacher to imitate her schoolfellows and learnt a strategy: drawing a large base-line (the desk) and then lining up the object. For Giusi the notion of a base-line is a very recent acquisition and it is still necessary to become more confident with it. Other pupils are already questioning about the different sides of objects and may distance them from the frontal drawing (1). So, Greta (5) refers to a previous strategy: drawing things where they are held, where there is place for them without paying attention to their position. But the levels of generalization are different: somebody speaks about ***houses*** *(and* *the house was indeed the most problematic object to draw because of its two sides that were seen by many pupils and that provoked a conflict between the stereotype - rectangle - and the perceived shape), somebody speaks about* **things**. *But some things are just houses (3), because of the choice of the other terms (fronts).*

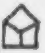
(b) In the previous discussion some pupils had begun to introduce terms of the mathematics register (line, angle, side, point of view). Some weeks later the teacher raised a conceptualization discussion: What does point of view mean? *The pupils again referred to the particular experiences (the scene), that constituted a kind of implicit shared context, but tried and used a generalizing language speaking even more of things and generating new examples. They succeeded in buildig collectively an elementary meaning of the term* **point of view** *on the basis of individual and collective experiences. The statements that were institutionalized by the teacher (picking up the pupils' statements) were the following:*

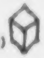
- The point of view is the position from which we observe;
- Changing the position modifies the things that are seen;
- Changing the position modifies the part of a thing that is seen;
- If two people go in the same position, they see the same things;
- The problem of points of view had been the motive for the scene game.

(c) At the beginning of the 2nd grade, we planned to create a conflict situation. Real-life drawing was well established: the pupils had drawn several objects and sets of objects from different points of view and compared the drawings in balance discussions. Nearly all had learnt to line up the objects (we had no more objects scattered over the whole sheet), to interpose objects that were not distinctly visually segregated and to eliminate hidden lines (we had no more "transparent" objects). Complex sets of objects partially interposed were represented as projections orthogonal to the sheet. We thought of presenting a very simple object: one cube in front of each pupil with an edge in the foreground. The marked geometrical structure had to induce conflicts between the three-dimensional shape and its two-dimensional representation, because of the difference of world and sheet frames of reference. The task was: draw the cube from your point of view, just as you see it. *We obtained some different products that were summarized as follows (the numbers of solutions are added):*

A (4) 

B (1)

C (8) 

D (2) 

The balance discussion was very interesting. The very beginning episode is the following:

Teacher: Well! Let's look at how it did go! Have you succeeded in drawing the cube: let's go..

Ingrid: I ..the thing that I have..yes..I've asked myself..I've said: What do I not see? And then I've drawn what I saw and not what I didn't see and then it worked out.

Elisa: At the begining it was so difficult. I've made as Ingrid and I've succeeded. But very very slowly..it was so difficult.

Greta: Er..I've followed the angles so I was able..then..at first I've said, I've asked myself: Have I to draw also what I do not see or what I see? Then I've read what was written and I've realized that we had to draw only what we saw.

Marco: I too, as Ingrid, I've asked myself: What is that I do not see? Then, after I've asked myself, I've drawn first the upper side and then I've begun the bottom side.

This short episode is an evidence of **internalization** *of previous balance discussions where the focus in comparing drawings had just been in the questions: What did you really see? Then the pupils explained the strategies, focusing on* **internally-oriented gesture**. *They contrasted the drawing C and D. D had been obtained by only two pupils but it was sufficient to provoke a conflict. So the pupils had to go into the difference between the geometrical properties of a three-dimensional shape (whose bottom side is flat) and the properties of its representation (whose bottom side is pointed). The base-line appeared a true* **obstacle**. *But some weeks later a similar task was proposed and a girl said:* Oh, now I **see** it differently!

In the school year 1990/91, the project has gone on for 3rd graders together with dissemination to some other classrooms. The former activity has expanded to a new activity. The motives have changed. The new motive for teachers is **the representation of the visible world by means of geometric techniques**; the new motive for pupils is **the analysis of images** (drawings, photos, works of art) and **the coordination of spatial perspectives**. The analysis of space representation in art has been considered an element of the early introduction to historical development of civilizations, that begins in 3rd grade. The early conflict between three-dimensional shapes and two-dimensional representations has evolved in giving to the latter a different cultural status: recent experiments show that pupils are becoming aware of the separation between experiential and representative space, of the possibility of making their relationships explicit and how geometrical objects only belong to the representative space.

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MEANING: IMAGE SCHEMATA AND PROTOCOLS

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Abstract: *The most prominent goal of teaching mathematics is that the learners construct meaning for the mathematical objects, terms, theories and the like. To attain this goal one needs sufficient knowledge and theories about how people individually and socially construct meaning, what kinds of cognitive mechanisms produce meaning and how the production of meaning can be supported by pedagogical and didactical measures. In this paper a theory of meaning is presented which is based on the concept of image schemata proposed by Johnson (1987) and Lakoff (1987). Meaning is viewed here to be induced by concrete "mental images" as opposed to propositional approaches. Protocolling one's own actions in specific situations and processes is proposed as a possible means to develop adequate image schemata. Related conceptions can be found in Dörfler (1988, 1989a, b).*

On formalism

There can be no doubt that mathematics as a field of human knowledge and as a human activity has many and even disparate facets. It is therefore neither possible nor sensible to address the totality of what it means to know or to do mathematics, the more so, since this is changing in accordance with historical, social, cultural and economic conditions and contents.

The focus of interest here will be on mathematical concepts and on how subjects can come to understand a concept, how they can construct meaning, how they can make operative use of concepts and the like. This is motivated by the well documented lack of meaning on the part of a high percentage of our students at any age level. Again, meaning and understanding are not one-dimensional phenomena, neither subjectively nor objectively. A clear indication of this are the various philosophical and epistemological theories and modes of interpretation of what is the meaning of mathematical concepts (like the formalist, intuitionist or platonist approaches). For sure, there is the possibility of formal meaning and formal understanding of mathematical concepts. Such a meaning is based on certain undefined basic objects and (a finite set of) rules (axioms) for handling them, for operating with these objects. Examples are the real numbers in the usual axiomatization or the axiomatic treatment of geometry. It is nowadays well accepted that such a formal meaning in principle suffices to establish a mathematical object (like a group, a vector-space) as long as the rules for operation are free of any contradiction. (I do not refer here to the problems evidenced by the results by Gödel.) There are even examples from history (for instance the complex numbers) when there was a phase during which only a formal meaning (i.e. objects plus operating rules) was available. Also the widespread intellectual resistance against a purely formal meaning (like for the complex numbers) is well known. On the other hand, Euler has operated with many ideal elements (in the sense of Hilbert) only based on formal rules obtained by extrapolation. Generally speaking, it appears to be a highly developed or advanced level of mathematical thinking if one is able to use operating rules in a formal way and to study them in a likewise formal manner. It is a specific point of view, a deliberate decision where to focus one's attention and what to make use of. This "formal attitude" has to be acquired and developed, it has to be experienced to be sensible and advantageous for certain purposes. All the following is therefore not to be understood as an argument against the formal aspects of mathematics, rather I view them as something which should be attained in the learning process but which should neither be its starting point nor its sole goal.

There are many advantages of a formal approach because of using well defined operation rules. One obtains high security and accuracy and, above all, the formal thinking in principle can be mechanized and automatized, it can be taken over by machines like the computer. But it might be, the prize for that is a loss of imagination and creativity.

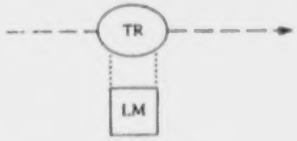
Image schemata

In this section I want to offer a non-propositional approach to how meaning of concepts - as constructed by the individual - can be grasped theoretically. Thereby, meaning is to be understood as the subjective meaning for the individual. In other words, the question is what guides the individual at the socially correct or adequate use of concepts (as designated by words in some kind of language)?

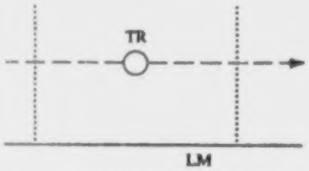
A word of caution is appropriate here. I think that it is impossible to comprise all that which constitutes meaning and understanding for the individual in one single type of theoretical description. First of all, we have no, and can get no, hold of what "meaning" is. There is no apriori fixed phenomenon which is referred to by the term "meaning" and would be amenable to empiric investigation and theoretical description. Further, one should also refrain from the view that there is a given objective meaning (of concepts, words, terms, theories) which the subjective understanding by the individual is to approximate as closely as possible. Who should or could be the owner of this objective meaning and how is it expressed? What we want to do is to explain sensible (i.e. socially accepted) cognitive behavior and these explanations necessarily have a theoretical character. Further, there it will be necessary to develop differing, mutually exclusive but each other complementing theories. Under certain circumstances, cognitive behavior will appear to be of one kind, and under different ones, of another. If even a comparatively simple thing like the light has to be described by complementary theories (wave-particle complementarity) how should one expect a single approach to suffice for the complexities of human cognition. Different theories will then be more or less suited to explain and/or predict observable cognitive behaviors.

I want to stress that the theoretical approach presented here points to a rather neglected direction but is needed to understand various cognitive and linguistic phenomena.

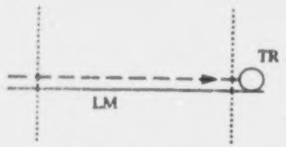
I will take as starting point a theory developed by Johnson and Lakoff. They use the ter image schema to denote a (hypothesized) cognitive mechanism which regulates the u of words (and other linguistic units). An image schema thereby is a schematic structu which in a highly stylized form depicts or exhibits the main features and relationships situations and processes to which potentially the word refers. More precisely, (part o the meaning of a word is generated by a kind of family of such image schemata whi are related to each other by processes corresponding to metaphorical and/or metonym transformations of meaning. Lakoff presents a detailed analysis of the meaning of t highly polysemous word "over" as it is observable in its use by competent speakers. "chained family" of such image schemata serves as theoretical explanation of this behavio Here are some examples of those image schemata (Lakoff, pp. 419-434).



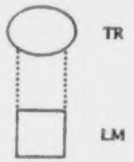
The plane flew over.
Schema 1



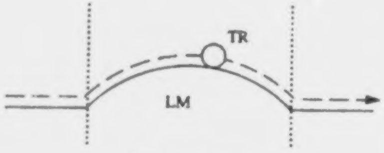
The bird flew over the yard.



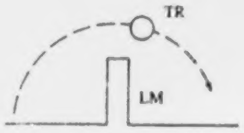
Sausalito is over the bridge.



Hang the painting over the fireplace.
Schema 2



Sam walked over the hill.



The dog jumped over the fence.

Some remarks are appropriate here. These drawings are not to be equated with the image schemata but they are a means for talking about the latter, for "representing" them in the scientific discourse. Therefore, it is not asserted that we have a cognitive store of such drawings which we then use when speaking, arguing, listening etc. The point is rather that our understanding of words shows features as if it were based on the interpretation, application, projection, transformation etc. of geometric structures of this sort. The image schema is then to be viewed as consisting of these cognitive activities of interpreting, applying, projecting, transforming etc. of the geometric-figurative schemata. In fact, when asked to describe their understanding of a word like "over" subjects refer either verbally or by gesturing to schemata which resemble those drawings.

The cognitive mechanism "image schema" works as if certain parts, constituents (like points, lines etc.) in the drawings serve as variables which can be instantiated in many admissible ways by concrete objects ("application" of the geometric schema). In the case of "over", for instance, this enables one mentally to correctly posit objects which are talked about and related to each other by "over". In a sense, the image schema permits to derive (from the discourse) a concrete mental image of the situation reported about which again can be depicted externally and be used in the further discourse. Shifts of meaning can be interpreted as shifts of the focus of attention on parts of the geometric schema, for instance, to the endpoint of a movement.

The main lesson we want to learn from the theory of image schemata and their empirical usefulness is that one can assume that meaning (subjectively) is not solely of a propositional and linguistic character but is based essentially on mechanisms which appear to be explainable by being objective-concrete and geometric-schematic. An image schema is to be understood as a (hypothetical) cognitive process which is based on highly schematic templates (paragons, models or the like) for the referential meaning of words. I would add here to Lakoff's theory that the individual derives those templates from the experienced and interpreted use of the respective linguistic term in the social discourse. Therefore, image schemata are acquired and learned by the bodily experience of the individual and present some relevant features of this experience to the individual. They are therefore related intimately to perceptions and movements and present those of their features which are constructed as important by the individual.

A final point to be stressed is that image schemata are used to make relationships (like those meant by "over") cognitively manipulable and understandable. The corresponding geometric schemata do not depict any object but serve to express relationships which are of spatial or temporal character.

This is now the point where I want to turn to mathematical concepts which too stand for relationships and complex systems of relationships. Without ruling out the possibility or even necessity of propositional and verbal presentation and understanding of mathematical topics I will try to substantiate the following thesis: The cognitive manipulation of mathematical concepts is highly facilitated by the mental construction and availability of adequate image schemata. In other words, the subjective meaning of mathematical terms has a non-verbal, non-propositional and geometric-objective component. The individual understanding of a mathematical topic possibly is best grasped as a kind of interplay between the propositional expressions and corresponding image schemata. This thesis

is to counter the prevailing verbal, propositional teaching of mathematics at all leve (though to a lesser degree at lower levels). This dominance of the verbal presentatio mode is convincingly demonstrated by the current text-books for schools and univers ties. Despite this dominance there is some evidence in the literature that the learn mostly unconsciously and implicitly constructs some kind of image schema. In sever papers (e.g. Vinner 1983) the term "concept image" is used which appears to point into similar direction though "concept image" admits propositional components. And surprisingly, more often than not it will be this image schema which determines the cognitiv behavior, and not the propositional definition of the concept. Remarks of a similar vei are made by Davis (1984).

We had characterized an image schema as the cognitive interaction with, and interpretation of, a kind of geometric schema. The latter one can be externalized and depicted as geometric drawing or even as a material model. By doing so, also the cognitive interactio with, and manipulation of, these schematic models will become more explicit and conscious. It is here where I want to expand the theory of Lakoff to apply it more smoothl to mathematical thinking and understanding. With Lakoff the image schema appears t remain implicit and not consciously reflected as a kind of automatic cognitive mechanism Here I want to view an image schema as the perceptive and/or cognitive interaction with and manipulation of, some kind of object-like model, be that a material one, a drawing o just imagined. I will call the latter the (concrete) carrier for the image schema. Man (but not all) of the well-known so-called representations for mathematical concepts potentially can serve as carrier for a related image schema. But let me stress, the image schema is exclusively a cognitive process in the cognizing individual and the "representations" are more or less suited for stimulating that process. Therefore, an image schema as such cannot be shared with anybody else, only the carriers can be communicated and in some cases the pertinent cognitive manipulations have a corresponding material way of manipulating the carrier. The image schema is just the specific way of viewing, interpreting, using, transforming etc. the carrier. Therefore it is absolutely misleading to regard the concrete carriers to "represent" the respective concept. It might be much more appropriate to say that the individuals by cognitive relation to, and their activity with, the concrete carrier present the concept to themselves, make the concept present, cognitively and mentally. In regard of the respective (mathematical) concept, it is the cognitive and perceptive activities with the carrier which constitute the meaning of the concept for the individual.

As a consequence, the "same object" (this term, of course, can be criticized) can be the concrete carrier for very different image schemata corresponding to different concepts. Conversely, the same concept usually admits a variety of image schemata based on different carriers. It depends on which properties, relationships, proportions, transformations etc. are viewed to be relevant, on which of them the attention is focused, which of them are constructed by the individual. But once constructed, all that very often is experienced to be a property inseparable of the carrier. Image schemata are such individual or even idiosyncratic. A kind of objectifiation and standardization occurs in the process of social communication (negotiation of meaning), and subjective deviations can there be corrected. Further, there is a supply of socially normed and standardized carriers, and the teaching (of mathematics) should essentially guide the construction and development of appropriate image schemata by the learner.

Image schemata in this sense not only constitute meaning (of the respective concepts) for the individual but serve also other cognitive functions. Neither memory nor cognitive operations (like arguing and logical inferences) appear to be based solely on linguistic means but have a concrete basis in the form of appropriate image schemata and their carriers. Possibly, language is rather the means to talk about one's image schemata, and thinking can be viewed as the cognitive analysis and transformation of image schemata. Johnson-Laird (1983) has developed a theoretical model which very well explains the behavior of subjects solving syllogisms. His "mental models" very closely resemble what I have described here as image schema. What is very important: the conclusions are not obtained by applying formal rules but by manipulating carriers of image schemata!

Making explicit, and cognitively accessible, the concrete carriers of image schemata for mathematical concepts (or objects) will have a very far reaching consequence for the learner. By that for him there are available adequate referents of the concepts and the mathematical language can be used to talk about them, to communicate about them. Note that, according to Lakoff, the image schemata regulate socially sensible linguistic and other cognitive behavior! I would add, also in mathematics this kind of concrete regulation and guidance of thinking is of great importance. One can observe that for many mathematicians and mathematics educators the true referents for mathematical concepts are so-called "abstract objects". Yet, it remains very opaque and unclear what those objects are like. In any case, they are neither perceptible nor tangible, and teaching should lead the students to grasp those objects. The many material "representations" are viewed as materializations of the abstract objects. These "representations" are used as a means, as a pathway to the genuine mathematical objects. In platonist words: the representations are just the shadows of the true objects, of the mathematical ideas. The students rightfully have the feeling that they have no access to these mathematical objects and such they remain restricted to the formal, rule-guided manipulation of symbols without referents. The impression is created that behind the mathematical language and its terms lurks a realm of abstract objects accessible to the gifted and talented but locked to the others. But this is highly unfair and discouraging for most of our students. My thesis is that it is much more appropriate and encouraging to offer an accessible realm of objects as potential referents which can give rise to the individual construction of image schemata for the mathematical concepts.

By a further remark I want to highlight another difference of the position held here to how teaching and learning of mathematics is commonly structured. A basic, but often implicit, assumption is that mathematical meaning can be built up in a cumulative way by starting from elementary "particles". The more complex concepts obtain their meaning from being constructed out of building stones whose meaning is already defined or understood. An extreme case is the endeavour which the name Bourbaki stands for. The more difficult concepts are those which are made up from the greater number of "elements" and they have to come later in the curriculum. In my view, meaning has (besides the structural aspect) a predominant wholistic aspect which is not to be divided up into more elementary particles. This wholistic meaning corresponds to the pertinent image schemata for the respective concept. To attain it, first of all suitable concrete carriers (i.e. adequate fields of experience) are necessary. Thus I assert that by adequate didactical means much of the so-called higher level mathematics is directly approachable and the traditional systematic bottom-up approach could be avoided or partly circumvented. Of course, this does not mean that the same formalizations are possible but in a sense equivalent conceptualizations (e.g. of the derivative or the integral) are possible and sensible. For a related position see Tall (1990). It should be mentioned that also in linguistics one finds a related elementarism with respect to the semantic meaning of words and terms. Lakoff by his theory strongly opposes this theoretical position which is among others unable to explain in a satisfying way phenomena like prototype effects, metaphors or metonymies.

As a final remark: this approach to meaning and understanding is quite incompatible with Piagetian stage theories for cognitive development. The formal level is, for me, not a higher one but one complementing other forms of cognitive activity like those based on image schemata.

In the following I will describe various types of image schemata. Thereby, membership in a type depends on the image schema as a whole and not on the carrier. Further, the types are extreme cases which commonly will be mixed up in specific cases.

Figurative image schemata

Here the main activity relating the individual to the concrete carrier, and such constituting the image schema, is of a perceptive character. There are no transformations or operations carried out with the carrier. The carrier often shows distinctive features, properties and relations which guide or attract the attention and perceptive activity of the learner such that there is a good chance for developing the appropriate image schema. For those people who have constructed the adequate image schema the objective carrier then in a sense "represents" the concept which corresponds to the image schema. The carrier can now be called a "realization" or "materialization" of the image schema (or concept). But this is not to be viewed as an objective fact; it is only a manner of talking.

The cognitive construction of a figurative image schema demands the focussing of the perceptive and cognitive attention of the learner on specific differences, features, qualities, relations etc. This can be accomplished by salient characteristics of the experienced or observed objects, situations or processes. Yet, in general the guidance of some kind of teacher will be necessary who points to the crucial and typical (for the intended image schema) characteristics, extends or delimits the attention (verbally or by gestures), corrects misguided constructions, etc. This guidance never will completely determine the construction since the latter necessarily is the activity of the learner. And it is exclusively her/him who discerns, who separates, neglects, combines etc. There is much empirical evidence for this highly personal quality of the construction of images schemata (see all the literature on concept images). Intensive social communication is needed to accomplish image schemata which comply with the established standards. This is a very strong argument against the highly individualized learning in the traditional school which offers little chance to the individual learner even for realizing her/his (unavoidable) misinterpretations and inadequate constructions. But the common way of teaching implicitly is grounded on a pedagogical realism which asserts the existence of a universal meaning which can be communicated by displaying appropriate exemplars for the intended concept or via verbal definitions. But, the objects (or drawings) and definitions do not carry with themselves any meaning, they do not "speak" to the learner. The failures of the students are then all too often explained as being caused by their inability. What is overlooked is that the failure often is just a mismatch with the social standard which needs adjustment by feedback from the others. Thereby we stipulate the human faculty for constructing images schemata out of experience. Yet, the latter necessarily is individual and has to be guided (not determined) appropriately.

Examples: Image schemata for geometric concepts (like circle, square, polyhedron) can be of a primarily figurative character even if this does not match the adequate understanding of those concepts. The common drawings are then concrete carriers but of course other "representations" may serve the same purpose, at least in principle. For concepts related to real functions like monotonicity, extremum, convexity, zero of a function, periodicity etc. very likely an appropriate figurative image schema will be of advantage when operating with those concepts. Yet, it is well known that there is some danger of misinterpreting the corresponding carriers (i.e. drawings of grpahs). Generally speaking, I think that for very many mathematical concepts an adequate image schema must comprise a figurative component which has to be complemented by operative, relational and symbolic ones (see the following sections). Examples abound in the form of "representations" which I would view as potential concrete carriers.

Operative image schemata

Figurative image schemata as purely perceptive schemata are an extreme case which rarely will occur. Mostly an image schema comprises operations with, and processes on, its concrete carrier. An operative image schema is so to say the other extreme: it is a schema of operations on/with its carrier. Is it for the figurative schema perceptive processes which

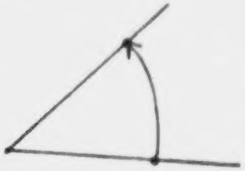
determine its essential content so now specific operations (transformations, productions etc.) with the carrier play this role. The carrier here again is objectual: either material, depicted or imagined. The carrier too, will be schematized and such lends itself specifically to the execution of the operations (material or mental) constituting the image schema. Already in the case of "over" some of the image schemata have to be viewed as operative ones if they are to present the intended meaning to the individual.

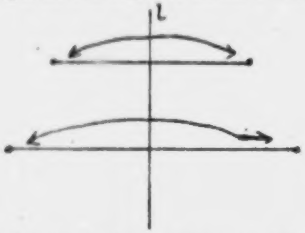
Essentially, all what has been said in general about image schemata and about figurative ones applies here as well. Foremost, I want to stress that also operative image schemata have to be acquired, to be constructed by the learner. For that, again structured guidance by a teacher is very important. What occurs rather often is that the students construct restricted figurative image schemata where a highly operative one would be adequate. This can be observed for instance with the graph of a real function where only an operative interpretation will give an adequate understanding. This is not to deny a certain complementarity of figurative and operative aspects but just that emphasizes the importance of either of them.

Example: The usual **cartesian graph** of a function needs an operative interpretation if it is to serve as the carrier of an image schema which can present the mapping and correspondence aspect of the function concept to its owner. This is a mental operation ("movement") of going vertically up from the x-value to the graph, turning left or right by 90° and continuing to the left (right) to find the $f(x)$-value. Similarly, mental operations are necessary to turn the other well-known "representations" of functions (arrow-diagram, table) into an appropriate image schema. Once more I should stress that neither of these "representations" does represent anything by itself, and the appropriate operative image schema has to be learned constructively by the student. The image schema operations will not be associated with one specific carrier but rather with a schematic carrier whose figurative outlook is irrelevant. Only by instantiating the variable components, the graph of a specific function will be obtained.

As another example we take the concept of an **interval** in the real numbers or of a **line segment** as a model for the continuum. I assert that this is not just a figurative image schema. The association of specific operations with the carrier (e.g. a drawn line segment) only gives an appropriate image schema. Among those operations is the unlimited divisibility (into equal parts) which then gives rise to such constructions as that of the decimal numbers. It is the (stipulated) executability of these operations which turns the figure of a line segment into the continuum. This executability is not a property of the concrete carrier but it is purely cognitive.

Image schemata for many mathematical concepts should comprise an operative component which complements the figurative aspect. For sure, geometrical concepts will have figurative image schemata (e.g. presenting the respective type of figure like triangle, square, quadrangle, etc.). But to gain fuller mathematical meaning operations like rotations and reflections must enrich the image schema (giving rise to the symmetry properties of the figures). It is quite interesting that geometric drawings on the other hand serve as carriers for the image schemata which present the very concepts of a rotation, a reflection, a symmetry etc. These image schemata, of course, are highly operative ones. The image schema for a rotation or for a reflection in a line might have as concrete carrier a drawing like this one:





It is quite illuminating to compare this with the carriers for the image schemata of "over" to get a better feeling how an image schema cognitively presents meaning to the individual. As a kind of condensed experience the potential operations get tightly associated with the carrier as its meaning for the individual. Very often, there remains no feeling of the constructedness of this association and the operative schema is then experienced as an inherent property of its concrete carrier. Similarly, the image schema is very intimately related to the word denoting it (like "over") such that for the speaker the subjective impression of built-in meaning of the word results.

To grasp the meaning of more complex concepts various image schemata will be needed. This already applies to "over"! The more it will be the case, say, for "fraction". For instance, there is the part-whole schema with its common concrete carriers. This is again an operative schema. Operations like composing the whole from the parts, dividing and sharing the whole into the parts etc. are needed to constitute the full and appropriate image schema for "part-whole".

In general, many of the common "external representations" can serve as the concrete carriers for respective image schemata. Here I want to put forward a critical point. In my approach, image schemata are a central mechanism for providing meaning and for guiding many cognitive and conceptual activities. Therefore, their development by the individual learner should be a main goal for the teaching of mathematics. This is quite contrary to the rather common view that "representations" are only a methodical means on the way to the genuine concept, to the abstract object. Due to this view the "representations" are often left behind, are only used in the first approach, but afterwards not referred to and worked with. After all, they are regarded as being only representations for something else which is the real thing and can be forgotten if the latter has been attained. In my opinion, there should be a continual and expanding use of the image schemata which comprises their being made explicit and conscious to the individual. This only would make possible the adjustment of the individual schemata to the socially accepted norms and standards

from which the former inevitably digress more or less. In the usual teaching by far not enough time and effort is devoted to the construction of image schemata which is left almost completely to the learner. I think this construction is a demanding task which can only be fulfilled in a social context with much feedback and communication. One relies too much on language as carrier of meaning and forgets what the language is about. A misleading catchword in this context might be the saying "mathematics is a language". I would add, yes it is, but it is a language about specific image schemata which lend meaning to the words and symbols. In any case, mathematics is not a purely formal language even if treating it as such sometimes is of advantage.

Relational image schemata

I have chosen this term to point to those cases where the essential content of the image schema consist of relations constructed at the concrete carrier. Those relations are not immediately perceivable but need to be mediated and constructed cognitively. This mediation often is supplied by certain transformations or operations (material or mental ones) with the concrete carrier. This is already the case with the various image schemata for "over" since there the mutual (spatial) relationships of the parts of the drawings (concrete carriers) are of central importance. These relations intimately correspond to movements like up or down which for instance have to be carried out with one's eyes when looking at the drawings. One does not need further explanation that many mathematical image schemata will be relational too. Consider the circle whose defining relationship – among others – corresponds to rotating a drawing of it in itself by an arbitrary angle or to rotating a line segment which is fixed at one endpoint. It is by these image schemata that a purely figurative understanding of "circle" is transcended.

Once more I stress that just showing a drawing (e.g. of a circle) will in general not suffice to stimulate the construction of an appropriate relational (or operative) image schema on the part of the learner. From the outset, too many relations (and operations) are possible and conceivable. Even if the student carries out a construction, the concrete carrier by itself is no guarantee that it will be the intended one. This again underlines the importance of the social guidance for the cognitive constructions.

Symbolic image schemata

Image schemata imply a specific view on their concrete carriers by regarding parts and elements as variable. Thereby image schemata obtain a generic quality which permits the presentation of the manifold and general through the specific. Thus, a wide range of concrete cases can be subsumed and understood by cognitive activities based on the image schema mechanism. Even a productive use is possible like the production and generation of new and hitherto unknown situations fitting an image schema. This generic faculty is

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even increased in those cases where the concrete carriers are symbolic terms. I think that formulas in mathematics can play that role of a carrier for an appropriate image schema. The latter then is made up by the spatial relations of the symbols in the formula and of the admissible operations and transformations with the formula. It is this image schema which lends meaning to the formula as its concrete carrier in an analogous way to how the image schema for "over" creates meaning and enables understanding.

Protocols of actions produce image schemata

We now turn to a central question: What is the origin and the genesis of image schemata? Johnson (1987) and Lakoff (1987) point out that many image schemata are based on bodily experiences and reflect movements and spatial relationships of the human body. In general, my thesis is that an image schema presents, to the individual who has it constructed, the schematised and generalised content of recurrent experiences. In this way the image schema permits to understand and organise future experience and to guide intentional activity. With respect to image schemata for mathematical concepts I propose a special mechanism which is hypothesized to lead to the conscious construction of image schemata by the individual. This is termed here **protocol of an action**, following Dörfler (1989 a, b).

This method of making a protocol of one's actions is mainly suited to generate an operative image schema for those mathematical concepts which are genetically related to certain actions or processes. But I surmise that some kind of operative approach can be developed for most mathematical concepts. On the other hand, a protocol is one possibly way leading to an image schema and it presupposes that the pertinent action is intentionally carried out, or at least observed, by the individual.

It is to be stressed that the protocol, like the image schema, is a cognitive process which produces as observable product a concrete carrier for the intended image schema. Let me then state some determining properties of a protocol given that some appropriate action is carried out (see the examples):

1. A protocol produces a structural system of perceivable and manipulable objects like icons and symbols but also linguistic signs. Through these objects and their spatial/temporal deployment are noted and (schematically) described the relevant and characteristic steps, sections, intermediate stages, results etc. of the carried out actions. It is important that the learner her-/himself produces a protocol of her/his own actions.
2. The protocol implies and presupposes a specific view at the action and results from a certain punctuation (division into sections) of the action process. Such, the protocol determines certain specific spatial/temporal relationships between the objects acted upon and those are constitutive for the intended mathematical concept (compare also Dörfler, 1987). In usual terms, one could say the protocol generates a representation of the concept. Because of the specific punctuation needed the didactical guidance of the learner by a (kind of) teacher is of utmost importance.
3. The produced structured system of signs gains meaning only through the protocol and as such has no meaning whatever. In other words, the generative relation to the action cannot be omitted. Further, neither is the protocol uniquely determined by the action nor the latter by the protocol. Therefore the relationship protocol-action has to be constructed and fixed by convention. Given this conventional relationship the schema of the action process can be reconstructed from the protocol (or its material product). In some cases the concrete basis of the protocol on itself permits the execution of actions which again produce the very protocol.
4. Since the protocol produces a highly schematic record or account of the action it can be the starting point for far-reaching generalizations. These will be accomplished by viewing certain parts (signs, icons etc.) in the protocol as variables which can be interpreted in many ways different from that in the original action. The protocols and their concrete carriers can themselves become the object of analyses, transformations and compositions thus giving rise to new actions (operations with the protocols) and protocols, i.e. to new mathematical concepts.

Examples of protocols

In the following I will list several mathematical concepts together with a sketch of how the model of the action protocols could be applied to them. These sketchy remarks are not meant as didactical programs but those could and should be developed starting from them.

Natural numbers. The action clearly is any kind of counting activity. Protocols may be the verbal uttering of the number words; using fingers or other representatives of the counted objects; lists of strokes. All these give a precise account of the main steps of the counting procedure with regard to the question "How many?". On the other hand, those protocols permit the replication of these essential steps on themselves – the protocols are prototypes of (sets of) countable objects.

Decimal numbers. Again counting activity is the starting point but now it is organized in a certain way by using counting symbols of the values 1,10,100,1000 and so on and by always changing to the next higher unit when possible. As in all the examples we (have to) assume that the learner is in a cognitive stage which enables him to carry out adequately the actions. For instance, here he must be able to form mentally higher order units which again can be counted. That granted, the resulting protocol will be something equivalent to the decimal representation of the number of counted elements. Clearly this representation reflects the main phases or steps of the "decimal counting" and in a very condensed form permits its replication (when assuming that all the higher order units have already been formed). A similar approach holds for other bases of the number system.

Decimal fractions. Consider a measuring process by which one wants to exhaust a quantity (say, less than one unit) by tenths, hundredths, thousandths and so forth of the unit. This process naturally leads to a decimal fraction as its protocol and result. By reading this protocol one can mentally reconstruct the process: first take d_1 tenths, then d_2 hundredths and so on until the whole quantity is obtained. Here, like in other cases, the protocol can further be used as an action plan for producing an object (a quantity) which just will give rise to the protocol when the actions are carried out on it. Again one can use different bases for the measuring process.

Fraction. A fraction can be viewed as a very curtailed or abridged protocol of a measuring process: m/n represents "m copies of quantity A are equal to n copies of quantity B" or "m copies of the nth part of quantity A give the quantity B". Again there is a wealth of different activities possible around this relation of m/n to measuring processes from which for instance concepts like equivalence of fractions, ratio of quantities, addition of fractions could be derived. This example makes very clear that a pattern of signs has to be explicitly related to the actions which constitute (part of) the meaning of the signs (as an action protocol). In this specific case the protocol m/n itself does not permit replication of the action but this gets possible if m and n are replaced by protocols (for the related counting processes). This corresponds to combining the counting and measuring processes. The number symbols can be viewed as names for the respective protocols (or for the actions themselves) and m/n then gets a compound protocol. Still, this example makes clear that producing and reading a protocol heavily makes use of signs whose meaning is determined by convention.

Matrix. Consider two sets A, B of elements (objects, properties etc.) and some kind of measuring the "value" of a relation between elements of A and of B. A protocol will very likely take the form of a matrix. A matrix such can be viewed as a way of noting the values of (binary) relations and the arrangement of its elements in lines and columns mirrors the stepwise process of determining these values. In this sense then a matrix is not just a static array of elements but it has inherently a dynamic structure or even several ones (e.g. scanning by lines or by columns); it is a schema of writing down symbols. Conversely, the idea of the matrix as the intended protocol determines the order of the steps in any measurement of this kind. In a very similar way any kind of table can be viewed as the protocol of some measuring (in a broad sense).

Graph. The drawing of a graph can in a way very similar to the previous example be considered as the recording of a relation on a given set of elements. "Reading" this protocol gives a full account of the (formal) properties of the respective relation, a weighted graph even reports on the strength of the relation between any two elements. As is well known, this type of protocol has become the topic of extensive mathematical investigations (Graph Theory).

Angle. Angles are mostly viewed as static geometric objects which are defined in one way or another. But they can also be used as the protocol of a rotation (around the vertex of the angle): the first leg is rotated into the direction of the second. The very same protocol then results from angles which usually have to be considered as equivalent. These equivalent (static, figurative) angles correspond to actions which very naturally everybody will view to be the same ones. Here again the action can be replicated directly on the protocol or its replication can easily be imagined.

Algebraic terms. These can be viewed as protocols recording some calculations or rather the schema of the calculations. Conversely, a term tells me which calculations are to be made in which order (according to conventions agreed upon). Tree–like representations can play an equivalent role. In my opinion it would be very helpful for an adequate understanding of elementary algebra if the concept of algebraic terms and their manipulations were based on such an understanding: Terms as schematic protocols of (my own) calculations which I can use to give an account of them to myself or to others. Like in all previous examples a term in this way permits to reflect on the calculation (e.g. to study the influence of certain numbers on the outcome).

Some research questions

This paper is a purely theoretical one and I am aware that the position developed here needs more empirical support beyond what is known by now. Some routes of investigation could be:

1. Empirical and experimental research about cognitive behavior which possibly can be explained by the theory of image schemata. I refrain from saying "research about image schemata" since "image schema" is a theoretical concept and as such not observable or measurable. Case studies like those carried out by Lakoff (e.g. with regard to "over") appear to be very instructive and informative. This would be in line with the existing research about concept images but focused on aspects of the interplay between image schema and concrete carrier. The question for short could be: What guides the mathematical behavior of students (of any age)? Can mathematical reasoning be explained as being based on propositional thinking (i.e. on definitions, theorems etc. in verbal form) on concrete–objective thinking (like image schemata), or on an interplay of those dimensions and possibly others?
2. Which features of potential concrete carriers (or their material models) affect the cognitive construction of an adequate image schema? In other words, how can one guide the perceptive and cognitive attention of the learner such that she/he is enabled to form the required relations and operations? What will be the role of a teacher in this process?
3. Which theoretical assumptions about specific image schemata and concrete carriers can be derived from the observed behavior of subjects?
4. Based on research results it would be an eminent didactical task to develop potential

concrete carriers. Of course, much is already available there but possibly needs to be investigated with regard to the development of image schemata.

5. A very promising direction of development is that known as (computer-)microworlds which might provide to the students the experience which then could be condensed to adequate image schemata. What is the influence of computer experiences and activities on the development of image schemata? See, for example, Dörfler (1989 c) and Tall (1990).

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ON THE STATUS OF VISUAL REASONING IN MATHEMATICS AND MATHEMATICS EDUCATION

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Visual reasoning plays a far more important role in the work of today's mathematicians than is generally known. Increasingly, visual arguments are also becoming acceptable as proofs. Cognitive studies, even though identifying several specific dangers associated with visualization, point to the tremendous potential of visual approaches for meaningful learning. Computerized learning environments open an avenue to realize this potential. It is therefore argued that the status of visualization in mathematics education should and can be upgraded from that of a helpful learning aid to that of a fully recognized tool for mathematical reasoning and proof.

INTRODUCTION

Visualization is generally considered helpful in supporting intuition and concept formation in mathematics learning. Fischbein (1987), for example, notes that "one of the characteristic properties of intuitive cognitions is immediacy. Visualization ... is very frequently involved ...". Similarly, Bishop (1989), in a recent review of research on visualization in mathematics education, concludes "that there is value in emphasizing visual representations in all aspects of the mathematics classroom". Two qualifications shall be added to these generally positive evaluations; one concerns difficulties with visualization, and the other concerns the status accorded to visualization in mathematics education, its epistemological value.

During the past few years, many student difficulties with visualization have been identified. These include students' inability to see a diagram different ways (Yerushalmi & Chazan, 1990), their difficulty to recognize transformations implied in diagrams (Goldenberg, 1991), their incorrect or unconventional interpretation of variation and co-variation in graphs

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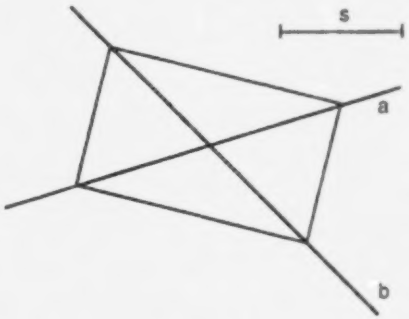
(Clement, 1989), their lack of distinction between a geometrical figure and the drawing that represents this figure (Laborde, 1988) and, most importantly, their lack of coupling their visualizations to analytic thought (Presmeg, 1986). These difficulties are all related to what Fischbein called "intervening conceptual structure": Diagrams and figures contain relevant mathematical information in a form that is determined by certain rules and conventions, which often are specific to a particular type of diagram. They are therefore not accessible to students who have not had the opportunity to get acquainted with these rules and conventions. Such difficulties with visualization have by now entered the consciousness of mathematics educators, if not of teachers. They will accompany but not be central to the argument in this paper.

The second qualification, and the one that will constitute the central concern of this paper, concerns the low status accorded to visual aspects of mathematics in the classroom. This is typified by the student who, after a detailed and lengthy presentation of a visual argument by the teacher, raises the hand to ask: "Can you also give a *mathematical* proof for this?" The reluctance of students to use visual reasoning has been reported profusely in the literature. To cite one typical source: "Despite the calculus teacher's predilection for diagrams, our research indicates that students resist the use of geometric and spatial strategies in actually solving calculus problems." (Balomenos, Ferrini-Mundy & Dick, 1988). More details on students' avoidance of visual considerations have been reported, for example, by Vinner (1989) and by Eisenberg and Dreyfus (1991).

A significant piece of evidence on the status of visual argumentation is constituted by various classifications of proofs that have been established by mathematics educators. For example, Blum and Kirsch (1991) classify "inhaltlich-anschauliche" (content-visual) proofs as pre-formal. The message is that visualization may be a useful and efficient learning aid for many topics in high school and college mathematics, but nevertheless an aid, a crutch, a step, sometimes a necessary and important step, but only a step on the way to the real mathematics. Such an attitude on the part of mathematics educators and teachers, whether justified or not, is bound to influence students to avoid the use of visual arguments.

This situation has unfortunate effects: it eliminates a versatile tool of mathematical reasoning for all students, and it may prevent some of the weaker ones from successful problem solving. In fact, Bondesan and Ferrari (1991) report that even poor problem-solvers adapt or invent new strategies in a geometric setting, but not in an algebraic one. And Presmeg (1986) has found that while children have little difficulty in generating visual images their imagery is predominantly concrete pictorial, with far less pattern imagery, and hardly any dynamic one. Since pattern and dynamic imagery is more apt to be coupled with rigorous analytical thought processes, this means that students are likely to generate visual images but they are unlikely to use them for analytical reasoning. In this paper, we want to make the point for precisely such visually based analytical thought processes or, for short, visual reasoning.

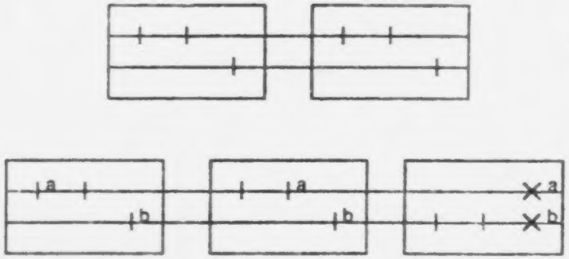
To make the idea of visual reasoning more concrete, consider an example taken from a unit on geometric loci, designed specifically for developing visual reasoning patterns (Hershkowitz, Friedlander & Dreyfus, 1991): Suppose you have to deal with the following problem: Given two intersecting lines in the plane, find the geometric locus of all points the sum of whose distances from the two lines equals a given length. This problem can be approached from many angles and in many ways. One global way of starting out would be to argue that the locus must be contained in a bounded region of the plane because any point that is very far away must be far from at least one of the lines. A more local way of starting would be to ask whether any points of the locus are going to lie on the given lines, and to start searching along these lines. This search may be approached dynamically by starting at the point of intersection and moving out along one of the two lines. As one does so, the distance from the other line grows from zero without bound, therefore one must at a certain stage pass a point which belongs to the locus. Because of symmetry reasons, this yields four points. The locus turns out to be the rectangle whose corners are these four points. This again is not trivial but needs a detailed analytical argument, which may be based on appropriate ratios in suitably chosen similar triangles. Every part of the above argument will be considered as visual reasoning because it makes essential use of visual information. Visual reasoning used in this kind of argument may be global or local, dynamic or static, but it is never purely perceptual: It includes valid analytical argumentation leading from step to step. The thesis of this paper is that such visual reasoning is very frequently used and accorded increasing value in mathematics, and that it would behoove mathematics education to follow suit.



VISUAL REASONING IN MATHEMATICS

Mathematicians are not innocent of the fact that visual reasoning has a low status. Many indicators point to the fact that most mathematicians rely very heavily on visual reasoning in their work. But with few exceptions, mainly in combinatorics and category theory, these same mathematicians do their utmost to hide this fact. Indeed, mathematicians tend to be secretive about their work; they tend to hide very carefully how they obtained their results. They present only the final finished formalized product. They do not let the reader see any of the processes. And many mathematicians behave the same way when they give a lecture about their work: The lecture will usually not be about the methods they used to obtain their results; diagrams will be used, at best, to illustrate an argument which, as the lecturer will point out, can be proved rigorously using an algebraic formalism.

There are few instances where mathematicians explicitly describe how they obtained their results. One of these is contained in a publication by van der Waerden (1954) on the topic of idea and reflection in mathematics research. In there, he uses as illustration a discussion with two colleagues during which they found a proof to the following conjecture by Baudet: If the set of natural numbers is split into two disjoint subsets, then at least one of the subsets contains an arithmetic progression of length L (where L is arbitrary). The report on their discussion takes up seven pages and contains eight figures with possible patterns for number sequences to be distributed into two (or more) subsets. The first figure is reproduced here. It is accompanied by the sentence "Wir zeichneten die Zahlen als kleine Querstriche ... auf zwei waagrechten Linien, die die beiden Klassen darstellen sollten." (We drew the numbers as small crossbars ... on two horizontal lines which were supposed to represent the two subsets.)



The entire argument rests on the patterns given in these figures. As van der Waerden states: "Der Beweis den ich im *Nieuw Archief voor Wiskunde 15*, 212 (1927) dargestellt habe, ist die genaue Ausführung des hier anschaulich erläuterten Gedankenganges." (The proof, which I have presented in *Nieuw Archief voor Wiskunde 15*, 212 (1927), is the precise execution of the line of thought presented here visually). The five page paper in which he published this proof does not contain a single diagram. I doubt, however, that many mathematicians are able to understand the proof in that paper without recreating van der Waerden's diagrams (or other, similar ones). Diagrams are essential for mathematical thinking, but their use is being hidden by mathematicians as best they can.

Other reports on how mathematicians think also point to the overwhelming importance of visual aspects; a systematic attempt to discuss mathematicians' research thoughts has been undertaken and reported by Hadamard (1945). Although he insists on individual differences concerning the manner in which mathematicians' thoughts rely on mental images, Hadamard concludes that they, very generally, use images and that these images very often are of a geometric nature. He recounts that when thinking, practically all mathematicians avoid not only the use of words but also that of algebraic or other symbols; they use vague images. In particular, Einstein wrote to Hadamard: "Words and language, written or oral, seem not to play any role in my thinking. The psychological constructs which are the elements of thought are certain signs or pictures, more or less clear, which can be reproduced and combined at liberty." (Hadamard, 1945, p. 82).

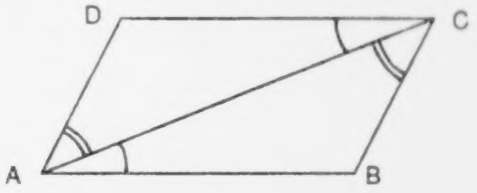
Why, then, do mathematicians hide their visualizations and the arguments based on them? Several reasons come to mind: Some, like Einstein's vague images, may never have become sharp enough to be describable in word or picture. Others, like van der Waerden's diagrams, have probably been judged unacceptable to the standards of mathematical publication common throughout most of the 19th and 20th century; these standards were strongly influenced by both, logicism and formalism. History shows that the standards have not always been so inimical to visual argumentation (Berra, 1986); and there is some evidence that the situation may be rapidly changing again. In the past few years, many mathematicians have addressed the importance of visual reasoning not only in discovering but also in describing and in justifying mathematical results. Rival (1987), for example, has written an article with the subtitle "Mathematicians are rediscovering the power of pictorial reasoning". The usefulness, even necessity of visual reasoning patterns in modern mathematical research has also been stressed by Devaney (1989). He recounts how he and three students have described certain dynamical processes through sequences of transformations in the complex plane, represented them graphically by means of computer programs, and then filmed these sequences. According to Devaney, the results of these rather time-consuming experiments have always been mathematically stimulating and many new mathematical results have been proved as an outcome.

Davis and Anderson (1979) go beyond stressing the power of visual reasoning for discovering new results in mathematics; they not only describe mathematics "done in actuality - as a series of nonverbal, analog, often kinesthetic or visual insights", but suggest that the "excessive emphasis on the abstract, analytic aspects of thought may have had deleterious effects on the profession". Among their examples is the Jordan Curve Theorem (A simple closed curve in the plane separates the plane into two regions, one bounded and one unbounded), which is visually obvious but whose analytic proof requires notions from algebraic topology and is therefore rarely presented even at the undergraduate level. Finally, and most importantly, Davis and Anderson refer to the existence of "purely visual theorems and proofs", and recommend to encourage the production of such theorems. Many but not all of these theorems have been found by means of computer-graphical support.

If, following Davis and Anderson, visual arguments are to be admitted as (parts of) mathematical proofs, the question naturally arises how (and even, whether) wrong visual arguments can be avoided. How often have we seen children rely on particular features of a diagram in a geometry proof, and thus present an invalid or at least incomplete proof? And although one would not expect mathematicians to fall into the same trap as tenth graders learning Euclidean geometry, some mistakes in visual arguments are far more subtle (see e.g. Blum & Kirsch, 1989, for a beautiful example), and it is unknown where the limits of such subtlety lie (if there are limits at all). Who is to judge the validity of a visual argument?

Three replies to this question will be given. First, that in many proofs visual arguments are unavoidable; second, that judgement of the validity of non-visual arguments is not safe either, and third that criteria for better judgement of visual arguments should be developed.

In a paper of philosophical nature, Stenius (1981) analyzes the epistemic function of the figure in a Euclidean geometrical proof. This proof is a modification of Euclid's proof that in a parallelogram opposite sides are equal. Given a parallelogram ABCD, in which AB is parallel to CD and BC to AD, prove that AB=CD (and BC=AD). The proof proceeds by drawing the auxiliary line AC (a diagonal of ABCD), and showing that the triangles ABC and CDA are congruent, from which the result follows. No doubt, this proof is quite typical for proofs in a Euclidean geometry class. And who could understand this proof, even if many more details were spelled out verbally, without imagining before the mind's eye a parallelogram? This already demonstrates that we use the figure in following the proof. But beyond this, Stenius asks, how do we know that BAC and DCA are alternate angles on parallel lines? Could not the point D lie inside the triangle ABC. The point here is not whether it could or could not but how we can know that without recourse to the figure.



Most teachers are well aware that they do use figures in proofs. But, many will answer, you are only allowed to use those features of the figure which are are not particular to the figure but generic; in our case, those which are true for all parallelograms, for which the particular drawn figure serves as a model. And who judges, asks Stenius, whether a figure can or cannot serve as such a model, and which features of the figure are generic? This has not been formalized, and if it had, the formalization would be quite useless to the first time student of Euclidean geometry. Therefore, the use of diagrams in teaching and learning Euclidean geometry must not be avoided, but quite the contrary, it must be analyzed and dealt with explicitly. More generally, correct mathematical reasoning with figures and diagrams should be stressed.

For those who, due to their formalist and/or logicist background, remain unconvinced and fear that proofs relying on visual reasoning are dangerous, because they depend on a substantial measure of validity judgement by mathematicians and mathematics teachers, let me just mention that the situation for sentential proofs is not different in essence, only in degree. In a wonderful dialogue between the ideal mathematician and an inquisitive student, in the book by Davis and Hersh (1981), the best definition of a mathematical proof (as opposed to a proof in formal logic) which the ideal mathematician comes up with is "A proof is an argument that convinces someone who knows the subject". In other words: the validity of the argument is judged by the expert - there is no machine algorithm to check a mathematical proof; and there is thus no a priori reason why some of the reasoning in a proof should not be diagrammatic or visual. Why, then, do mathematicians often object to visual arguments in proofs and why do they attempt to eliminate the visual reasoning before they publish a proof? This point is eloquently explained by Barwise and Etchemendy (1991). Although they agree that "we are all taught to look askance at proofs that make crucial use of diagrams, graphs or other non-linguistic forms of representation, and we pass on this disdain to our students", they claim "that diagrams and other forms of visual representation can be essential and legitimate components in valid deductive reasoning." They point out that mathematicians' expertise in judging the validity of linguistic reasoning is based on careful and lasting attention to this form of reasoning and that such a tradition and ensuing expertise for visual reasoning is lacking. Hence, they advocate and have begun a research program to devote similar attention to the judgment of the legitimacy of visual and mixed, heterogeneous reasoning patterns. Such a development is particularly important in view of the potential for dynamic visual reasoning offered by graphic computers; accordingly, their computer program Hyperproof plays a central role in this undertaking.

We thus conclude that the reasons for the attempts to minimize visual reasoning in proofs are not based on principle; in fact, visual reasoning is in principle even unavoidable; but the profession of mathematics, the experts supposed to judge the validity of the proofs, have neglected to develop their ability to carry out this judgement in the case of visual arguments.

To summarize, a clearly identifiable if still unconventional movement is growing in the mathematics community, whose aim it is to make visual reasoning an acceptable practice of mathematics, alongside and in combination with algebraic reasoning; according to this movement, visual reasoning is not meant only to support the discovery of new results and of ways of proving them, but should be developed into a fully acceptable and accepted manner of reasoning, including proving mathematical theorems. The availability of powerful graphic computers has played a non-negligible role in the emergence of that movement.

INDICATIONS FROM COGNITIVE SCIENCE

Few cognitive science studies deal directly with visual reasoning. Johnson-Laird (1988), for example, argues at some length for using mental models to explain inferential reasoning and then summarily states that "Vision yields mental models" (p. 231). Studies which do treat the effect of visual support on making inferences and solving problems, however, show that appropriate visual support has positive effects on students' understanding and problem solving. The following seem particularly relevant here.

In a review of more than a decade of work on the use of conceptual models for understanding, Mayer (1989) concludes that such "models will improve the ability of students to transfer what they have learned to creatively solve new problems"; the ability to creatively solve new problems is what Mayer terms understanding. There is little doubt that such understanding implies certain forms of reasoning, but this is not spelled out. Obviously, it is crucial for us to know what is meant by conceptual models. They are descriptions of systems from science, technology, programming and mathematics which spell out the major parts, states, and actions in the system; in each case, the model includes a pictorial representation of the explanative information, highlighting the key concepts and suggesting relationships between them. Although Mayer specifically includes text in his conceptual models, his findings show

that "illustrations help students organize information into meaningful mental models" and these, in turn, are at the root of their successful problem solutions.

Mayer did not explicitly identify the contribution of the visual component. Therefore the question may be asked, to what extent the visual features of his conceptual models were crucial. In support of this hypothesis, local studies can be found in which children's ability to answer inferential questions is clearly due to the pictorial presentation of information (e.g., Holmes, 1987). But with respect to solving problems in mathematics and science, the situation is far more complex: This is shown in a thorough analysis of some diagrams used to solve such problems: Larkin and Simon (1987) compare the accessibility of information needed to solve problems when they are presented in diagrammatic versus sentential form. The distinguishing feature is that diagrammatic representations explicitly preserve spatial relationships between components of the problem, whereas sentential representations do not. In diagrams, information is indexed by its location, thus giving the possibility to group all information about a single element together, and to express logical relationships spatially. Thus, diagrams do not necessarily describe spatial arrangements; they have inherent interpretations and conventions without which they are unintelligible. Those who know these interpretations and conventions can develop visual reasoning patterns exploiting the advantages of the diagram. Those who have not learned to read the diagram, cannot take advantage of it. Thus diagrams can be of enormous help in solving problems, but only to those who have had the opportunity to learn what the many, possibly complex features of the diagram represent, how they need to be interpreted and related among each other. Larkin and Simon have thus given precise expression to Fischbein's "intervening conceptual structure" mentioned in the introduction.

Finally, in his paper in this volume, Dörfler (1991) has expanded Lakoff's idea of image schemas as a theoretical basis for generating meaning in mathematics learning. As he states, for very many mathematical concepts, an adequate image schema must comprise a figural component which has to be complemented by operative, relational and symbolic ones. The objectual carrier for the figural component will often be a visual representation of the concept; the associated operative components enable visual reasoning with and about this concept. Dörfler's theoretical framework is thus not only compatible but fully resonant with Hadamard's description of mathematicians' thinking patterns, and consequently with most of what has been said heretofore about visual reasoning.

IMPLICATIONS FOR MATHEMATICS EDUCATION

Quite a complex picture thus emerges about the potential and the effective role of visual reasoning in learning mathematics. Theories and analyses from cognitive science clearly show the potential for an extremely powerful role for visual reasoning in learning many mathematical concepts and processes. A warning is, however, associated with this promise: Visual reasoning is based on expertise - it will be unhelpful if not impossible for the uninitiated. The promise made by cognitive science appears to be borne out by mathematical research activity: experts make extensive use of visual reasoning during the creative process; in addition, there is an emerging movement to give legitimacy to visual arguments also in the presentation of mathematical results.

Mathematics educators seem to have recognized the potential power and the promise of visual reasoning; but in spite of this, implementation is lagging: Students tend to avoid visual reasoning. The slowness of educational change in general may be one reason for this. But two additional weighty reasons are suggested by the above description: Firstly, while visual reasoning enters curricula and is even presented by teachers in the classroom, it is often given an air of an introductory, accessory or auxiliary argument, precisely because the experts, be they mathematicians, curriculum developers or teachers, do not assign full value and status to it. And from this attitude, students soon conclude that they don't really need to know and use the visual arguments.

Secondly, visual reasoning is difficult; it needs to be acquired by hard reflective work. Failures and disappointments are likely to result from unreflected, careless or too rapid introduction of visual representations. A wide variety of visual representations are used in mathematics: Geometrical figures, trees, set-theoretical (Venn) diagrams, number lines and Cartesian planes, often with function graphs or scatter plots, to name a few of the more institutionalized ones; sketches, diagrams and figures created locally for dealing with a specific problem may be even more important. There are some common features to all these, namely that figural entities represent elements of a conceptual structure and that spatial relationships such as proximity are used to express significant relationships between these elements. But differences between different types of visual representations are enormous: Some are pictorial, they directly represent objects, e.g. geometrical ones such as in the locus example in the introduction; most represent the mathematical objects under consideration in a symbolic, stylized manner. Some are static, others are dynamic or at least imply the idea of a dynamic transformation. The same mathematical object may take different shapes in different visualizations: A function for example may be represented by its Cartesian graph, or it may be represented as a point in some point set of functions or again it may be represented by means of the values of some parameters in a parameter space. Hence, the patterns of reasoning which are appropriate and useful in different visual settings vary considerably; different ways of reasoning need to be constructed for different types of visual representations; and each potentially harbors its specific learning problems. Cognitive characteristics and reasoning patterns applicable to large classes of visual representations may exist, but have not been researched so far. There are plenty of questions and few answers.

In order to give our students the chance to profit from and to appreciate the power of visual reasoning we, as a profession, need to upgrade the status of visual reasoning in mathematics; in our own mathematical thinking, we need to generate visual arguments, to learn how to examine their validity and to accord them parallel weight as we accord to verbal and formal arguments. In order to overcome students' tendency to avoid visual reasoning we, as teachers, need to use it not only frequently and consistently in searching for problem solutions but also at crucial junctures of our mathematical justifications with the aim to make appear both, the full power of visual reasoning and the importance accorded to it by the teacher. We need to give our students many opportunities not only to visually solve problems but also to discuss valid and invalid visual arguments; finally, we need to give our students full credit for correct visual solutions. In order to be able to do all this, and to make it permeate into teacher education we, as researchers, need to considerably expand our understanding of the cognitive and mathematical processes involved in visual reasoning. Detailed, content specific knowledge about the mathematical and educational validity of visual representations and reasoning patterns needs to be obtained for many different mathematical notions and processes; this includes the investigation of limitations, difficulties, obstacles and possible misinterpretations associated with the proposed visual representations.

Visual reasoning obtains its clearest expression if no alternative is available, if some mathematics is presented in purely visual form. Several developments in this direction have been proposed recently, some have been carried out and a few have been systematically implemented in classrooms. Two which are rather explicit in their reliance on visual reasoning will be briefly described here as exemplary.

Artigue (1989) has developed and taught a university level curriculum in which suitable computer software is used to help students develop a qualitative, geometric approach to the properties of solutions of differential equations. This qualitative study of differential equations is based on reasoning with functions which are not given explicitly by a formula, but only by means of information about their derivative(s). One of the explicitly stated aims of the curriculum was to lead students to work with curves without the support of a formula; in other words, to infer graphical information about the curves from graphical information about their derivatives. In order for this aim to become realistic, a complete break with the usual, formula based treatment of function at the high school level had to be made. Some of the phases in the curriculum were to get acquainted with basic notions such as slope field, isoclines, solution curves, and symmetries, to produce curves in a dialectic interplay between prediction and justification, and to learn about higher level graphical notions such as branching and flows including the variation of the type of flow in equations depending on parameters. One of the conclusions of the experiment was that, once the break with the habitual, purely algebraic approach had been effected, students accessed the geometric framework with relative ease, due to the fact that the complexity of their tasks was reduced by the possibility to use appropriate software.

Goldenberg (1989) conjures the vision of a radical restructuring of the pre-college mathematics curriculum, centered on an introduction of fractal geometry in junior high school. He proposes to "adopt a visual and experimental type of mathematical inquiry and learning", in order to "foster the development and use of qualitative, visually-based reasoning styles"; among these, he specifically includes visual proofs. He illustrates how problem posing may originate at the visual level and shows that questions about trigonometric relationships, about limits, about series, and about iterated processes arise naturally out of the detailed investigation of the geometry of fractal curves, their perimeter, border, enclosed area etc. The corresponding mathematical notions "are approached in visual, concrete, informal and intuitive fashions, with formal tools acquired as they are needed." In particular, a concept of function is apt to develop that is not only more general than the one usually developed at high school level, but also more robust and flexible. The entire approach is conditioned on appropriate software tools, that give the students freedom to explore the geometric objects under consideration by changing parameters and variables, including basic shapes and recursion rules.

Other projects based on purely or predominantly visual reasoning have been designed, among others, on feedback systems (Janvier and Garançon, 1989), plane geometry (Yerushalmi & Chazan, 1990), geometric loci (Hershkowitz, Friedlander and Dreyfus, 1991) and linear programming (Shama and Dreyfus, 1991). It is no accident that in all these projects, computerized learning environments play a major role. We will conclude this paper with some remarks on the potential and the problems arising in the use of computers for visual reasoning.

COMPUTERS AND VISUAL REASONING

Computers make it possible to represent visual mathematics with an amount of structure not offered by any other medium. Graphic computer screen representations of mathematical objects and relationships allow for direct visual action on these objects (rather, their representatives) and observation of the ensuing changes in the represented relationships; moreover, the situation can be inverted: it is possible to also investigate the question which actions will lead to a given change in the relationships. The result of such action often can be dynamically implemented; actions can be repeated at liberty, with or without changing parameters of the action and conclusions can be drawn on the basis of the feedback given by the computer program. The power of the computer for learning visual reasoning in mathematics derives from these possibilities.

Several projects have used the above considerations and exploited them in the development of software to achieve and investigate specific learning goals. To mention but a few examples: Tall (1991) reports on using the computer to encourage visually based concept formation in calculus; specifically, local straightness rather than a limiting process is suggested as a basis for developing the notion of derivative; Tall stresses that the goal is not only to provide solid visual intuitive support but to sow the seeds for understanding the formal subtleties that occur later. This implies that the students learn to reason visually with the details of screen representations of concepts such as function, secant, tangent, gradient, gradient function, etc. Kaput (1989) has used concrete visual computer representations to build on natural actions in the students' world with the aim of supporting the learning and application of multiplicative reasoning, ratio and proportion. In particular, he aims to tie the visually concrete and enactive operations on objects on the screen with more formal and abstract representations of these operations. Thus students' visual operations are directly used in the learning process. Yerushalmi and Chazan (1990) have given students the opportunity to empirically generate visual information about geometrical constructions and to infer conjectures from such information. Again, this cannot be done without visually based action (to generate the geometric information) and visually based cognitive activity to infer a conjecture. Shama and Dreyfus (1991) have used computer screen presentations of linear programming situations to allow students to develop their own solution strategies. For this purpose also, students need to analyze the problems in terms of the visually presented information and thus to give a visual basis to their strategies. All of these projects thus aim for detailed analysis of the relationships contained in the visual screen presentation and for reasoning based on such analysis.

In computerized learning environments it is possible to directly address and overcome some of the problems associated with visualization, mainly those related to lack of flexibility in the students' thinking; it is also possible to transfer a large measure of control over the mathematical actions to the student; but the potential of computers for visual mathematics does not by itself solve the more important problems which were mentioned in the introduction. In every case, visual representations need to be carefully constructed and their cognitive properties for the student need to be investigated in detail. The adaptation and correction of features of these visual representations on the basis of student reaction to them is an integral part of the development, and in some cases has been reported in the literature: Tall's choice of local straightness rather than a limiting process for the derivative is a case in point. Similarly, Kaput describes how he has found dissonances between students' visual experience and the semantic structure of the situation being modelled and has consequently designed a way to avoid such difficulties. These difficulties associated with visual representations can be overcome, but only if they are systematically searched for, analyzed and dealt with. In this endeavor, the design of student activities within the learning environment plays at least as important a part as the design of the computerized environment itself (Dreyfus & Halevi, 1990).

Little has been said in this paper about two important topics: verbalization and multiple linked representations. Verbal argumentation in mathematics suffers, to a large extent, from similar problems as visual reasoning. My insistence on visual reasoning should by no means be construed as an argument against verbalization - quite the contrary. There are in fact some indications of positive interaction between the visual and the verbal (Bondesan & Ferrari, 1991). Moreover, many of the cited examples do link the visual representations to algebraic ones and thus open the possibility for integrated visual/algebraic reasoning; I have consciously downplayed those aspects because the purpose of this paper was to make the point that visual representations and visual reasoning in mathematics must not be considered as a crutch for those who can otherwise not make the step to the "real mathematics". I have attempted to show that visual reasoning in mathematics is important in its own right and that therefore we need to develop and give full status to purely visual mathematical activities. Although I pressed one point, namely the visual one, the final goal, obviously, is not to be one-sided, neither on the algebraic side, nor on the verbal side, nor on the visual side. One goal is balance, as has been stressed already by Davis and Anderson (1979); and we should aim for more than balance: we should aim for integration of visual, verbal and algebraic thinking. Before one can aim for integration, however, one needs balance; and in order to achieve balance, visual reasoning needs to be given equal status and attention as algebraic reasoning.

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THE ACTIVITY THEORY OF LEARNING AND MATHEMATICS EDUCATION IN THE USSR

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The basic statements of the Activity Theory of Learning which has been developing at the Psychology Department of the MSU are formulated. Learning is analyzed with the emphasis laid on its central component, i.e., the action that is to be mastered by the learner. The structural elements and four functional parts of an action are distinguished. Three types of an action's orienting basis are described. The principles of organiziation of education in accordance with the Theory are mentioned. Examples of mathematics education are given.

The Activity Theory of Learning which has been developing at the Psychology Department of the Moscow State University states that social experience assimilation must be considered as mutual activity of the learners (learning) together with the teacher (teaching). Learning is the main component of the mutual activity, because its immediate outcome is an increase of learner's experience. Teaching creates conditions that favour learning. Its main functions are the following: creation of the motives of learning, presentation of information that is to be assimilated, control and correction of learning process.

The specificity of this approach is determined by very detailed analysis of learning activity. This activity includes the actions which are to be mastered by the learner together with the appropriate knowledge. Such actions and elements of knowledge vary in cycles of learning activity.Moreover, the learner performs also acts of proper learning. Their contents are relatively constant in various subject-matters.

Although the both types of learning actions have been analysed, the most carefully explored are its central components, i.e., actions that are to be mastered by the learner, and the corresponding knowledge. The merit of such exploration belongs to the late professor P. Ya. Galperin, the author of well-known Theory of Stage-by-Stage Development of Mental Actions, that was created in the 1950s and constitutes the nucleus of the Activity Theory of Learning developed later. Acad. N.F. Talyzina, who is the nearest disciple of P. Ya. Galperin contributed much to the theory; so did their numerous colleagues.

The theoretioal analysis lays emphasis on an action as the main unit of human activity. Structural elements which constitute the action process by their interaction and its components were described; the system of action's parameters and their values transformations in learning process were described. Learning itself was represented as a succession of definite stages producing those transformations as their outcomes.

We shall consider in brief the principle statements of the theory. The structural ("substantial") formative elements of an action are the following: its subject, its object, its means, its external conditions and its final product; action's procedure is a special characteristic. Subject presupposes the following substructures: motive, objective, ability in doing this action and so called orienting basis of this action. We shall define only the last concept.

Orienting basis of an action is subject's knowledge about 1) the concrete values contingents (sets) of initial elements of the action, i.e., object, media, etc. (with the exception of its product), which allow to realize the objective of the action; 2) the executive procedure variants; 3) the correspondences between those concrete values contingents and the concrete particular types of successful executive procedures. If such knowledge is true it constitutes the foundation of appropriate ability.

The process of action consists of four parts: orienting, executive, control and corrective ones. Orienting part produces knowledge of actual situation, i.e. of concrete peculiarities of initial scructural elements of the action. This knowledge allows to come to a decision concerning the mode of performing the main, trasforming part of the action, e.i. its executive part. The latter one must result in the desirable product. However this product, as a rule, turns out to be different from the planned one because the subject makes mistakes in the process of orienting; besides the external world produces perturbative influences. Control part of an action is designed for watching its process and comparing the result obtained with the planned one. If they do not coincide corrective part must be performed. These four functions can be represented also by an action as a whole and even by a special activity: orienting,executive,control and corrective ones.

Any human action can be determined by several parameters; each of them possesses some qualitative or (and) quantitative indices. The main psychological characteristics are the following: form of an action, level of generalization, degree of reduction (contraction) and mastery

An action's form characterizes the level to which it is inherent in the subject. Three major forms of action are distinguished: material, that of external speech and mental.

Material form presupposes operating on real objects by means of material instruments (e.g., a child may be engaged in learning to count material things by moving and grooping them with his hands). There exists a modification of material form which is called materialized one (e.g., a learner may count circles or sticks drawn in a textbook with the help of his finger). Material form has special advantages in revealing action's contents to a learner - its component operations, the sequence in which they are executed, etc., as well as in controlling the execution of each operation by a teacher.

External speech form is characterized by the fact that object action's is presented in the form of oral or written speech, and the action itself is performed as verbal reasoning. In such a form actions become theoretical (ideal) but not yet inaccessible to external observation and control. It should be noted that an action in the verbal form is not a verbal action, i.e., its aim is not communication, but obtainment of a new result (knowledge) by its subject.

An action in the mental form is carried out within one's mind, it presupposes operating on sensual images and concepts. Such an action is already entirely internal.

Besides, there exists an intermediate form of an action- a perceptive one (again, not to confuse

with a perceptive action!). It means operating on sensual images of things placed in subject's sight (e.g., visual count of objects).

Generalization of an action characterizes the degree to which those properties of the object that are essential for carrying out a given action are distinguished from others that are non-essential. The second way to determine this characteristics is to find out the proportion of the number of situations in wich the subject is able to fulfil the action to the total number of situations where successful performance of the action is possible. An example of a low level of generalization is the case when a child being able to count sticks in a visual form can not still count houses (" because they are too big").

Reduction of an action is determined by the relative volume of elementary operations that remain in it at a certain moment of its evolution as compared with the initial (maximal) volume.

Mastery indicates the measure of ease with which an action is carried out, the extent to which it is automated and the rate with which it is performed.

Besides these four parameters the Theory distinguishes a certain number of secondary characteristics wich depend in their values on the former ones, i.e., their indices change in the process of action's improvement as consequences of transformations of some primary parameters. The secondary parameters include reasonableness of an action, its conscious character, its abstractness, durability of appropriate skill, etc.

The parameters described, as well as the functional parts of an action, characterize not only the action which is to be mastered but also the proper learning action, i.e., the "generating" action that incorporates the former one.

However, due to permanent trasformations which the first type action undergoes in the learning process, taking into account their characteristics indices is a matter of special importance. Indeed, the process of learning presupposes their stage-by-stage transformation.

This Theory distinguishes six different stages of action development. In the first one must be created motives necessary for reception by the subject of a learning task and fulfilment of the activity designed. There exist various motives for learning: to escape punishment, to get some reward, self-affirmation, to be a good professional; besides, the very process of cognition may have motivating force. The latter type of motivation provides very powerful stimuli for learning. Like any other human occupation, learning activity is polymotivated, and it is the first stage that is to create an effective system of cognitive motives.

The second stage provides receiving by the learner the preliminary acquaintance with subject-matter material, more precisely, with a new action. He learns the most significant orienting points. In other words, the outcome of this stage is his understanding how to perform the action and what will be its result. However, true and complete assimilation can only occor if the learner performs the action by himself. This necessity is realized at further stages. Each of them provides transformations of the action in all the parameters with achievement of definite indices for every parameter.

In the third stage an action is to be fulfilled by the learner in material or materialized form. All its parts- orienting, executive, control and corrective - must be exerted in the most complete volume of elementary acts, i.e., with maximal spread. The both conditions provide real comprehension and assimilation of the action's contents by the learner; the teacher is able to control the learning process in detail and undertake correction necessary. In order the action becomes generalized the learner must do tasks which comprise all the typical cases of action's application.

Nevertheless, at this stage the learner must not solve too many tasks of the same type, otherwise, the action may grow intimely reduced and automated, thus the transition into the next stage will be difficult. During that stage the preparation should also take place to provide the transition to the nex stage. Fur this purpose, from the very beginning the material form of action is combined with the verbal form: the learner formulates verbally everything that he carries out materially.

The fourth stage aims to provide transformation of an action's form into external speech. At that stage the action experiences further generalization. It is still, however, unautomated and spread. Now speech begins to perform a new function: it not only points to the objects which are dealt with, but becomes an independent embodiment of the entire process. Such actions must remain spread: all their elementary operations are to be mastered in an external speech form. Generalized action acquires new properties at that stage. New typical situations may be introduced that could not have existed at the preceding stage. In developing the action of identification an object with a concept, e.g., generalizations were limited to two situations: 1) the object does belong to a given class; 2) it does not. At the fourth stage situations with incomplete conditions may be added, where correct answer must be the following: "It is impossible to learn whether the object belongs to the class". At the end of that stage an action begins to be carried out with omissions of some elementary acts, i.e. reduction occurs. At the fourth stage, however, an action should not be done automatically.

At the fifth stage an action takes the form of unvoiced external speech (" external speech to oneself"). It differs from the preceding one in that the action is performed as being expressed within oneself. However transformations of the parameters values now go more intensively. As soon as the action has reached the inner form it rapidly grows reduced and automated.

The last, sixth stage of an action's evolution presupposes its performance in internal speech. Very soon it becomes automated and stereotyped; it can not be visible from outside, it is inaccessible even to self-observation. On the one hand, it is an action in the inner form, on the other hand, as a matter of fact, it is already an act of thought: its process is hidden and only its outcome lends itself to awareness. Thus mental actions which are so unlike the external action on material things that generated them, are products of this latter's gradual transformation.

The stages described provide transformation of all parameters' indices, hence, every stage is to be characterized by concrete values of each of them. Unfortunately, it is only the parameter of form that allows to discern different qualitative states (they were mentioned above); the rest of the parameters' states remains rather vague in their qualities till now, though their quantities are measurable.

It should be noted that an action's evolution is not completed by the end of the sixth stage. It

undergoes significat changes that occur in the process of its execution together with many other external and internal actions. Its process, however, hides entirely from the observation and becomes inaccessible to direct control, neither to external, nor to internal one.

A matter of great concern is to decide what type of action's orienting base be used. It has already been noted that actions include four parts: orienting, executive, control and corrective. The orienting one plays a decisive role in the development of actions. It determines both the rate at which they develop and their quality. Orienting part is the process where the subject's knowledge of action's conditions -otherwise, its orienting- functions. Orienting basis may be characterized by several features:

1) form of its presentation (material, materialized, verbal);

2) its contents (complete,incomplete, redundant) and degree of generalization of orienting points (concrete, generalized);

3) mode in which learners acquire that knowledge (given to the learner in a preestablished form, developed autonomously by the learner). Experimental studies showed that form of presentig of orienting basis is of small significance for action's effectiveness, but characteristics of its contents and modes of acquiring it by a learner are very important ones. That is why the typology of orienting basis can only be built with taking them into account.

Thus, an orienting basis may be complete (sufficient), incomplete or (and) redundant with respect to any essential features. It may be presented in a particular form being of use for only one concrete action, or in a generalized one which allows to carry out a certain class of actions. A learner receives an orienting basis from his teacher as ready for use or finds it independently. Further, independent building of the orienting basis may be achieved either by trial and error method in the action's process itself, or by using a general procedure of its construction. General method may be got from the teacher or discovered by the learner independently. Each of those characteristics of orienting basis (completeness of the orienting points' set, their generalization and independence of acquiring them by learners) in the most simple case has two values: "0" or "1", i.e., existence or absence of the quality. Hence, there are eight (2^3) types of orienting basis. By the way, one can also take into account other characteristics and consider more than two states for each of them. Then in general case there may exist n^m types of orienting basis. Given limitations of those values are due to their practical importance and difficulty of consideration of intermediate values. However, experimental education has lead to further restriction, and the number of orienting basis types was reduced to three. Those three types are recognized as the most interesting to the Theory and practice of education. Now we consider them briefly.

Orienting basis of the first type has incomplete set of orienting points existing in a particular form and is found by the subject independently trial and error method.

Skill development requires much time, the learner makes very many mistakes during the process of learning. If any of the conditions changes the action usually becomes wrong.

Orienting basis of the second type is characterized by complete system of orienting points which are also presented as particular ones; the learner receives them from the teacher. Such an orienting basis provides unerring execution,the skill may be assimilated quickly and proves to be stable when action's conditions change.Nevertheless,transfer to new situations is rather poor.

The third type orienting basis contains the complete set of orienting points presented in over-all form that permits operating on a certain class of objects. In each concrete case a learner composes orienting base independently,using,however,the general method given by the teacher. *)

Skills can be developed rapidly and practically without errors in the learning process. The actions are resistent to conditions change; they can be easily transferred to new situations.

Learning with the first type orienting basis is of minimal effectiveness; it is typical for usual school education. On the contrary, the third type provides its maximal effectiveness. Moreover, it is adequate to the requirements that the society raises to human activity. Unfortunately, the third type presupposes significant expenditure of time and labour necessary for reorganizing the subject-matter information, searching for invariant units and appropriate actions' procedures to be realized by the learners.

So, in this respect the second type of orienting basis is more acceptable: as a rule, it is easier to determine a particular concrete system of orienting points. This type may be used successfully when there exists a necessity to provide rapid and effective acquisition by a learner of some specific material.

Activity approach to investigation of psychics presupposes that sensual images and concepts are united with a certain operational system in an entity. Action are not only the manner of their existence; ideal images and concepts can only be generated in an activity process. Even "mechanical" learning by heart requires performing a certain activity-mnemonic operations. But it is essential what activity does the learner perform. The quality of actions depends upon their object contents very much. If a concept is to be assimilated, then learning by heart its definition is not an adequate activity: the learner should rather perform acts of identification, deduce inferences, compare objects, classify them, etc. The more various actions underlie a concept, the better: it grows more fundamental and can serve as a powerful instrument of thinking. Secondly, the contents of the orienting basis of those action should include the essential features of the objects, otherwise, the knowledge will be superficial. Finally, the quality of knowledge is influenced also by indices of the parameters of the actions which underlie it, orienting basis type and the measure of inclusion of the knowledge into the subject's vital activity.

Unfortunately, actions necessary for knowledge assimilation are not always ready by the beginning of its process. As a rule, knowledge undergoes the stages that were described above, together with appropriate actions which provide transformations of their form and other parameters.

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*) In principle a learner can produce not only a particular orienting basis, but also a method that permits constructing various particular orienting bases. However it should be considered rather as creative activity than assimilation.

We have presented here the most significant statements of the Stage-by-Stage Theory. They characterize the psychological mechanism of acquiring social experience by human learner. Organization of an educational process on the base of appropriate knowledge is factually constructing a certain didactic system which should provide the desirable changes in learner's experience. We can not here present the entire system of principles which follow the statements mentioned.

Instead, we shall only list the main moments.

First of all, objectives of the learning/teaching activity must be formulated in a concrete form, in order the teacher should be able to establish whether they are realized or not. All the subobjectives must also be presented in detail, as some concrete activity components that are to be mastered by the learners. In each case their procedure must be determined as well as their psychological properties which were mentioned above. It is necessary to select subject-matter information and organize it according to the second or the third type of orienting.

As to organizing the process of learning a special attention should be given to control over learner's activity. It is necessary to find out whether the learner performs the planned action, whether it is correct, whether its form is adequate to the learning stage, what are indices of generalization, of reduction, etc. Control may be systematic or episodic (occasional): external or internal (with feed-back to the learner); it may take into account every elementary operation or only the final product. If the final product is merely controlled the information received may prove to be insufficient to determine necessary correction for errors that may be caused by different reasons. That is why the activity approach considers operation-by-operation control as more preferable. Similarly, systematic control is in the most cases more effective than occasional one. However, when choosing the type of control, it is necessary to take into account the concrete situation. In the beginning of material (materialized) and external speech stages control must be carried out over every operation and every action. Further, mere control over final product may be sufficient (and it is even more preferable than detailed one). In the latest stages the learning process must be controlled only occasionally and on the learner's initiative. External control dominates in the initial stages; further it is the learner who realizes the control functions.

The Activity Theory have been verified in numerous experimental investigations using both schoolchildren and adult students. The experimental study undertaken by I.A. Volodarskaya on education to performing initial geometric affine transformations dealt with poor schoolchildren. The invariant was found by the author that underlies every case of particular transformation. The following properties are characteristic of all affine transformations: a) translating a point into a point; b) reciprocal uniqueness, i.e., translating each point into one and only one point and two distinct points - into two distinct points; c) there exists a rigorously defined ratio between any two points in the initial object and the corresponding points in the final one. Moreover, the process of performing all those transformations includes the some components: the initial object; the final object; the object in relation to which the transformation is carried out; it is necessary to perform rotation around a point (axis, plane) or (and) shifting to a vector.

The peculiarities of the transformations which were presented in the orienting basis of the actions to be mastered by the learners determined their procedures. General prescription for transformation performing and particular prescriptions for each of its components were also made. The results of the actions were fixed by the learners in a special table that contained the names of the components; further they were omitted. It was only the orienting part that underwent form changes: the executive part always remained material. In the material action stage the learners worked on models which permitted them to realize the logics of transformations. In the process of carrying out the actions the learners could see that the components of the transformations were common; when varying each of them they obtained a number of particular kinds of transformation.

Similar reorganization of subject-matter material one can also find in the works done by G.A. Butkin who has analysed a general method of proving elementary geometrical theorems; by G. Nikola and N.F. Talyzina who have elaborated an over-all method of solving arithmetical problems; by T.V. Gabay who has programmed appropriately Fortran IV education and I.G. Shamsutdinova who has used the system analysis as a modification of the third type of orienting in probability theory course. All the authors have obtained high quality of knowledge, strong cognitive motivation and economy of time.

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Applications of R-Rules as Exhibited in Calculus Problem Solving

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Abstract

Seven wrong-rules (R-Rules) were identified in a study of student applications of calculus theorems to solving problems concerning various aspects of the notion of the derivative. All R-Rules are distortions of well known theorems in three modes of distortion.

The Problem

Recent research has shown that students lack basic understanding of conceptual calculus problems.(Orton 1983, Tufte 1988, Seden 1989). Future instruction of calculus should put a greater emphasis on conceptual understanding and application to problem solving. (Steen, 1987).

In order to adapt instruction accordingly, it is essential to identify the causes of the lack of understanding and analyze the resulting difficulties in solving conceptual problems .

In a former research, (Amit & Vinner,1990), the understanding of the derivative was examined. Some misconceptions were identified, to which the failure in solving problems might be attributed.

The present paper presents a further step in the study of the derivative. In particular, our goal is to shed light on how students use or misuse theorems in solving conceptual problems.

With respect to the application of mathematical theorems in problem solving, Fischbein & Kedem (1982) and Vinner (1983) found that instead of applying a general statement which has already been proven, to a particular case, high-school students repeat the general proof in terms of the particular case.

Movshovitz-Hadar et al (1986) found that high school students applied mathematical theorems in a distorted form in solving algebra, trigonometry and geometry problems. They identified two kinds of distortions: 1) distortion of the antecedent of a familiar theorem and 2) distortion of the consequent of a familiar theorem.

This paper presents distortions of calculus theorems as students applied them to "conceptual" problems.

The Method

Population: Participants were 294 first year university students from the faculties of economics, natural sciences, and technology who participated in a mandatory calculus course. Most students had a strong high school mathematics background , including some introductory calculus topics. As a whole the students had a high score on a scholastic examination which is used for university admission.

Sources of data: Data were collected from written answers to questionnaires administrated to all students, followed by personal interviews with 30 students.

Each questionnaire included 10-13 conceptual problems which students had to fully justify their answers. The score for each problem was based both on their solution and justification. These questionnaires were used a as mid-term examination with their scores on it counting for 20% of their final grade. Students, threfore, took it seriously and were interested to present their knowledge.

The instrument: Out of the questionnaire the following three problems were selected to be analysed in this paper. The selection was based on types of distortions of theorems that were exposed in their solutions.

Problem 1:

$f(x) = |x-6|$. What is the derivative of f at x=6 ? Justify your answer.

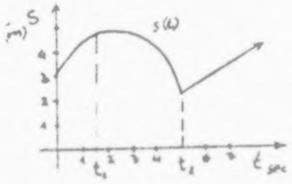
Problem 2:

The following figure describes the position of a particle moving on a line at time t.

In answering parts a and b, make your best estimation and justify it.

a) What is the velocity of the particle at t_1?

b) What is the velocity of the particle at t_2?



Problem 3:

$f(x)$ is defined over the closed interval [-3,4] as follows:

$$f(x) = \begin{cases} -x^2+4 & x<2 \\ 3x-6 & x\geq 2 \end{cases}$$

a) At what values of x is the tangent to the graph of $f(x)$ parallel to the line through A(-3,-5), B(4,6) ?

b) What are the minimum and maximum values of f over the interval [-3,4] ?

The Method of Analysis

Student solutions were analysed both in a quantitative manner - to determine the level of performance and in a qualitative one - to determine the nature of performance. In this paper, only the latter is discussed. The qualitative analysis identified patterns of errors which were then analysed and interpreted in a manner called "constructive analysis" (Movshovitz-Hadar, et al, 1987, pp.5). The aim was to determine how students distor theorems to obtain wrong rules. Specifically we asked: "To what rule, can the wrong solution be attributed ?".

Such rules, that were constructed from the wrong solutions given by the students are called : **wrong rules**, abreviated **R-Rules**. The R-Rules presented below are distortions of well known mathematical theorems:

The Intermediate Value Theorem (I.V.T.), and its corollary concerning zeroes of a function. The Mean Value Theorem (M.V.T.), the Extreme Value Theorem (E.V.T.) and the theorems concerning extreme values and critcal points in a closed interval (E.C.T.).

Results and Analysis

The following wrong solutions are not anecdotes. They represent many wrong answers that were very much alike.

Each wrong solution is followed by a short analysis and an R-Rule, the application of which can explain this solution. (The wrong solutions and explanations are presented in an exact translation from Hebrew).

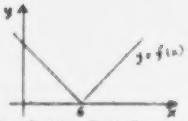
Problem 1, solution 1.1: "The derivative at x=6 is zero, f'(6)=0. Justification: For x<6 the slope is -1, therefore f'(x)=-1. For x>6 the slope is +1, therefore f'(x)=+1. Hence, at x=6 the derivative passes from a negative value to a positive one. According to the Intermediate Value Theorem (I.V.T), at x=6 the derivative is zero".

Analysis: The students distorted the antecedent of the I.V.T. by ignoring the necessary requirement for continuity of the function. In this case,they refered to the derived function which does not even exist at x=6.

Such an error could arise from applying R-Rule no.1:

R-Rule 1: If is any function defined on [a,b] and f(a) <0< f(b), then there exists a number c between a and b such that f(c)=0.

Problem 1, solution 1.2: "The derivative at x=6 is zero. f'(6)=0. Justification: At x=6 the function possess its minimum value (see the figure), therefore the derivative is zero there."



Analysis: The students distorted the consequent of the E.C.I. theorem, by ignoring the possiblities to have an extremum at non-differential points. From the fact that $f(6)$ is the minimum of f, they concluded (wrongly) that the derivative there is zero. This may be interpreted as the application of R-Rule no.2 :

R-Rule 2: For any function f, if f(c) is the minimum or maximum of f, then $f'(c)=0$.

Problem 1, solution 1.3: "The derivative at $x=6$ is zero. $f'(6)=0$.
Justification: Let's take the interval [0,12]. The function is continuous there, so according to the Mean Value Theorem,
$f'(6) = (f(12)-f(0))/(12-0) = (6-6)/12 = 0$."

The above answer was clarified in an interview:

q: Why did you consider the interval [0,12] ?
a: Because it is symmetric.
q: What do you mean by "symmetric" ?
a: You have the same distance from both sides of 6.
q: Yes, ..
a: six is in the middle of [0,12] so you can for sure use the M.V.T. for six, you have enough space from both sides.

Analysis: In this solution, the M.V.T. was distorted twice. In the first distortion, the requirement of differentiability on (a,b) was ignored. Although continuity on [a,b] had stayed intact,as mentioned explicitly in the solution, the function is not differentiable at $x=6$. This ignorance is exhibitted in R-Rule no.3:

R-Rule 3: If f is continuous on [a,b] then there exists a number c between a and b such that $f'(c)=(f(b)-f(a))/(b-a)$.

The second one was a distortion of the consequent of M.V.T. Students fixed the number c (from M.V.T.) exactly in the middle of the interval [a,b]. As demonstrated in the excerpt above, they chose the interval to be [0,12] because in that case the derivative $f'(6)$ was exactly at the mid point of the interval.

The relevant R-Rule is:

R-Rule 4: If f is continuous on [a,b] and differentiable on (a,b), then a number c exists, exactly in the middle of (a,b), such that $f'(c) = (f(b)-f(a))/(b-a)$.

Problem 2a, solution 2.1: "The velocity at t_1 is the derivative there. The derivative at t_1 can be calculated by M.V.T,. Given $t_1=1.5$ I'll take 1.5 units from both sides of t_1 (which has to be the middle) and substitute it in the formula :
$f'(t_1) = (f(3)-f(0))/(3-0)$."

Analysis : Here, like in the former solution, the interval [0,3] was fixed in such a way that $t_1=1.5$ will be in the middle. We see that even when the problem is in real life context - meaning, instantaneous velocity is required and not just "a naked derivative", and even though the function is smooth over (0,3), unlike solution 1.3, students applied the above R-Rule 4. The use of R-Rule 4 was repeated again and again in problems where data about the function were presented in number tables and the task was to evaluate an instantaneous rate of change.

Problem 2a, solution 2.2: "As a matter of fact, the velocity at t_1 is the derivative there, given by the formula:
$f'(t_1) = (f(5)-f(0))/(5-0)$.
Justification: the function is continuous on [0,5] and differentiable on (0,5), and the M.V.T. can be used. So, there is a point between 0 and 5 (in this case t_1), for which the above equation exists".

Clarification by interview:

q: What do you mean by "in this case t_1"?
a: It means: find the derivative at t_1, then put it into the formula of M.V.T.
q: Do you have t_1 in that formula?
a: No; I have c inside the interval, and I change c for t_1.
q: Is it legal ?
a: Sure, if you have continuity and differentiability then c exists, so you can choose any t_1 to be c, as long as t_1 is inside the interval".

Analysis: the consequent of M.V.T. was distorted by choosing an arbitrary number c. The specific interval (0,5) was chosen because the requirements of the M.V.T. hold intact there. The number c was arbitrarily located at $t_1=1.5$. From the interview it becomes clear that any other point in (a,b) could be "used" as the number c (from the M.V.T). The R-Rule exhibited here is :

R-Rule 5: If f is continuous on [a,b] and differentiable on (a,b), then for any number c in (a,b) , $f'(c) = (f(b) - f(a))/(b-a)$.

Problem 2.b, solution 2.3: "The velocity (meaning - derivative) at t_2 is zero, because at a minimum point (like t_2) the derivative is zero".

Analysis: The distortion of E.C.I here, is identical to the one in solution 1.2 although here it is in a rate of change context. We can assume that the above R-Rule 2 was applied here with respect to the function presenting the motion of the particle.

Problem 3a, solution 3.1: "There is no point at which the derivative is equal to the slope of AB.
Justification: The function is not differentiable (at $x=2$) and the requirements of the M.V.T. do not hold intact, therefore, the number c does not exist ".

Analysis: A wrong inference was exposed here.
Since the requirement for differentiability in the M.V.T. does not exist, students inferred that the consequent of the M.V.T. cannot exist. The translation of this inference into an R-Rule is the following:

R-Rule 6: If f is not differentiable on (a,b) then there is no number c between a and b such that $f'(c) = (f(b)-f(a))/(b-a)$.

Problem 3b solution 3.2: "The maximum value of f is 4, at $x=0$. The function f has no minimum.
Justification: $f'(x)=0$ only at $x=0$. $f'(x)=-2x$, $f'(x)=0 \Longrightarrow x=0$. $f''(x)=-2<0$, therefore at $x=0$ we have a maximum. There are no more zeroes of f', so there is no minimum of f ."

Analysis: Here, the consequents of the E.V.T. and the E.C.T were distorted. In the distortion of E.V.T. students denied the existence of extrema in a closed interval by claiming that "there is no minimum". In the distortion of E.C.T. students ignored the existence of extreme values in a closed interval at non-differentiable points or endpoints, and focused only on zeroes of the derivative. Taking $f(0)=4$ as "the maximum" of f instead of "a local maximum" of f could be considered as a mistake. But, the claim that there is no minimum because there are no more zeroes of f indicates that R-Rule no.7 may have been applied:

R-Rule 7 : For any function f, only if $f'(c)=0$ then f(c) is the maximum or minimum of f.

Note: R-Rules no.7 and no.2 together, are equivalent to the R-Rule **For any function f, f(c) is the maximum or minimum of f, if and only if $f'(c)=0$.** Yet, R-Rules no.2 and no.7 are applied in different situations therefore they are presented separatlly

Discussion

The R-Rules presented above, which are all distortions of mathematical theorems, can be classified into three types: (types 1 and 2 are based on Movshovitz-Hadar et al, 1986)

1) Distortion of the antecedent : Applying the claim within conditions which differ from those stated in the theorem.
2) Distortion of consequent : Ignoring or modifying the original claim while keeping the conditions intact.
3) Distortion of an inference rule : Applying invalid logical inference on given antecedents and consequents.

In the following table R-Rules are classified according to the type of distortion, the distorted theorems and the wrong solutions in which the R-Rules were exhibited. are mentioned:

R-Rule no.	Type of Distortion	Theorem	Solutions no.
1	antecedent	I.V.T	1.1
2	consequent	E.C.I	1.2, 2.3
3	antecedent	M.V.T	1.3
4	consequent	M.V.T	1.3, 2.1
5	consequent	M.V.T	2.2
6	inference rule	M.V.T	3.1
7	consequent	E.V.T, E.C.T	3.2

This study does more than identify wrong rules; it relates patterns of errors to particular R-Rules.
Defects in the learning process may lead to R-Rules applications. Some defects are listed below; others will be discussed in the session.

1) **"Red Herring" experiences**: example- Students (and textbooks) apply many times the Mean Value theorem to quadratic function where the number c in $f'(c)$ is always equal to the mean of the two chosen points a and b. For quadratic functions this is always the case, and students generalise this to other functions. Such experiences might cause the development of R-Rule 4.
2) **Fallacies in mathematical language**: There is an ambiguity between the universal quantifier and the existential one. For example, between statements like "for any number c such that..." and "there exists a number c such that". Such ambiguity might cause applications of R-Rules 4 and 5.

3) Invalid logical inferences: The statement: if p then q is often logically misinterpreted as equivalent to: if not p then not q. In analogy to mathematical theorems it means: if the antecedent does not hold then the consequent can not hold. This invalid inference might cause the false negation of theorems, as presented in R-Rule 6.

In Conclusion

In this paper, some wRong Rules which students probably applied in solving conceptual problems, were obtained from student solutions to conceptual calculus problems. Calculus instructors should be aware of the above wrong rules and strive to eliminate them. More research is needed to examine those R-Rules and to establish an appropriate treatment.

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EFFECTS OF DIAGRAMS ON THE SOLUTION OF PROBLEMS CONCERNING THE ESTIMATION OF DIFFERENCES

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Summary - The study was aimed to verify whether schematic pictures facilitate the solution of insight problems in which subjects were asked to estimate differences between two given states. The results of 6 experiments showed that pictures improved the performances of 13-14 year-olds in simple problem-solving tasks and that pictures aided 16-20 year-olds, but not 13-14 year-olds, in solving complex problems. In conclusion, the facilitatory effects of pictures occur only when subjects are able to realize the relationships between the visual patterns and the problem data.

Six experiments were carried out in order to study the heuristic effects of complete and partial schematic pictures - such as istograms and diagrams- on problem-solving. The variables which have been considered are: the characteristics and the degree of difficulty of the problems, the features of the pictures, and the age of the subjects.

In the present study insight problems, whose solutions require to realize the relational systems underlying the data, were always employed. More precisely, in such problems subjects had to estimate differences between initial and final states. It was hypothesized that pictures facilitate the discovery of the problem structure by simultaneously representing the whole problem field and highlighting its salient elements (Bejat, 1972; Frandsen and Holder, 1969; Kaufmann, 1979; Wicher et al., 1978).

EXPERIMENT 1

13-14 year olds were presented the following problems in which the transformation of an initial state into a final state is described (simple diacronic problems):

a) tie problem (Mosconi and D'Urso, 1974): "Paul and Peter are two friends who have the same number of ties. Paul gives Peter five ties. How many ties has Peter more than Paul?"

b) age problem (Raudsepp, 1980): "When Mary was 9 years old, her mother was 41 years old. Now Mary's mother is twice old than Mary. How old is Mary now?". Subjects were submitted to the problems in one of these conditions: without picture, with partial picture, with complete picture.

Responses are reported in these tables:

Tie problem	Without (N=26)	Partial (N=26)	Complete (N=26)
Correct response: 10 ties	9	8	19
Wrong responses	17	18	7

Age problem	Without (N=26)	Partial (N=26)	Complete (N=26)
Correct response: 32 years	2	3	8
Wrong responses:			
18 years	10	18	9
27 years	4	1	7
other	10	4	2

Results indicated that complete, but not partial, pictures improved problem-solving performances in both tasks (respectively: chi square test=11.45; $p<0.01$; chi square test=5.72; $p<0.05$).

EXPERIMENT 2

A problem involving a double transformation (complex diacronic problem) was used (Maier and Solem, 1952): "A man bought a horse for $70, sold it for $90, bought it again for $110 and finally sold it for $130. How much did that man earn in the business?".

The problem was presented to 13-14 year olds in one of these conditions: without picture, with partial istogram, with complete istogram, with diagram.

The following table reports the frequencies of the various responses:

	Without (N=20)	Partial ist. (N=20)	Complete ist. (N=20)	Diagram (N=20)
Correct response: $40	11	9	11	12
Wrong responses:				
$20	6	8	6	8
other	3	3	3	0

No significant effects of the pictures on problem-solving were observed (chi square test=0.95, n.s.).

EXPERIMENT 3

It was hypothesized that the ineffectiveness of the pictures on the Maier and Solem's problem was due to the complexity of the pictures. So, the same problem and the same pictures were presented to older students.

16-20 year olds received the problem in one of these conditions: without picture, with complete istogram, with diagram.

Solution frequencies are described in this table:

	Without (N=15)	Istogram (N=15)	Diagram (N=15)
Correct response:			
$40	9	13	5
Wrong responses:			
$20	1	2	7
other	5	0	3

Analysis showed that the performances in problem-solving of 16-20 year olds were affected by pictures (chi square test=8.88; $p<0.05$). More precisely, the istogram hinted at heuristic strategies of thinking; in contrast, the diagram produced misleading biases in reasoning.

EXPERIMENT 4

The aim of the subsequent experiments was to assess whether facilitive effects of pictures occurred also in problems describing differences between two simultaneous states (sincronic problems).

In the present experiment this simple sincronic problem was employed (Kanizsa, 1973): "A brick weighs a kilogramme plus half a brick. How much does a brick weigh?"

12-13 year olds were presented the problem in one of these conditions: without picture, with partial picture, with complete picture, with complete and stressing picture.

Subjects performed the task as follows:

	Without (N=15)	Partial (N=15)	Complete (N=15)	Stressing (N=15)
Correct response:				
2 kg	1	1	2	6
Wrong responses:				
1.5 kg	11	11	10	7
other	3	3	3	2

The complete and stressing picture increased the number of solutions of the problem (chi square test=3.09; $p<0.05$), while the other pictures had no effects.

EXPERIMENT 5

The experiment was designed to test the hypothesis that schematic pictures facilitate the solution not only of simple, but also of complex sincronic problems, such as this one (Mosconi and D'Urso, 1974): "An elephant and a mouse weigh together 2 tons and 100 grams. The elephant weighs 2 tons more than the mouse. How much does the mouse weigh?".

13-14 year olds were submitted to the problem in the same conditions of Experiment 4.

No significant differences in the performances of the four groups were found (chi square test=4.01, n.s.), as appears from this table:

	Without (N=15)	Partial (N=15)	Complete (N=15)	Enhancing (N=15)
Correct response:				
50 grams	3	1	5	5
Wrong responses:				
100 grams	9	13	8	9
other	3	1	2	1

EXPERIMENT 6

16-20 years old students were presented the elaphant and mouse problem in one of the following condition: without picture, complete picture, complete and stressing picture.

With elder subjects the same pictures, which were ineffective in 13-14 year olds, resulted in a degree of success in the problem-solving task (chi square test=5; p<0.05), as this table shows:

	Without (N=15)	Complete (N=15)	Enhancing (N=15)
Correct response: 50 grams	3	10	7
Wrong response: 100 grams	12	5	8

So, there is evidence that in complex transformation problems pictures facilitate solution processes in 16-20, but not 12-13, year olds.

GENERAL DISCUSSION

In conclusion, schematic pictures facilitated the solution both of diacronic and of sincronic difference problems because they made clear the structure of the problem field. However, if the meaning of the figural patterns was easily understood, facilitatory effects occured both in subjects of 12-13 years and in subjects of 16-20 years. In contrast, if pictures included conventional elements, these effects occured only in older subjects, because for the younger subjects it was hard to realize the relationships between the visual representation and the problem data.

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LE PROBLEME DU STATUT DU "MILIEU" DANS UN ENSEIGNEMENT DE LA GEOMETRIE AVEC SUPPORT LOGICIEL

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ABSTRACT

In this report, we use a piece of research carried out with junior high school pupils in a computer environment, in order to analyse how such a kind of environment affects what is called the "milieu" in the theory of didactic situations developed by G.Brousseau. We first give an overview of this piece of research. Then, after a theoretical introduction to the notion of "milieu", we analyse how some of its usual characterisitcs are modified by the computer environment used here, how the didactic system tends to react to these perturbations with the corresponding effects on pupils.
The research report will be given in English and an English version of the text will be available at PME.

I - INTRODUCTION

Cet article est issu d'une recherche concernant l'enseignement de la géométrie au niveau du collège dans un environnement informatique constitué à partir du logiciel "Euclide" que nous décrivons succintement dans le paragraphe suivant.
Cette recherche a débuté en 1988-89 en classe de quatrième (élèves de 13-14 ans) avec un double objectif :
- d'une part, étudier les potentialités offertes par le logiciel Euclide au niveau d'enseignement considéré et élaborer des séquences d'enseignement exploitant efficacement ces potentialités,
- d'autre part, exploiter la mise en oeuvre expérimentale de certaines de ces séquences pour étudier les modifications introduites par un tel environnement dans le fonctionnement du système didactique.

En fait, même si le support de la recherche est un logiciel précis, nous faisons l'hypothèse que le travail mené doit nous permettre d'identifier des phénomènes qui ne sont pas spécifiques de l'utilisation de ce logiciel mais ont une portée plus générale, et nous essayons de prendre en charge explicitement cette "décontextualisation didactique".

Nous présenterons globalement cette recherche dans le paragraphe III en nous centrant successivement sur les hypothèses faites au départ sur les potentialités du logiciel, sur les directions d'analyse privilégiées dans l'étude du fonctionnement du système didactique, sur les méthodologies utilisées. Nous nous focaliserons ensuite dans la partie IV sur les questions de "milieu" : après une présentation théorique de cette notion, nous essaierons d'identifier les perturbations apportées à son statut usuel par l'environnement informatique considéré ici, les stratégies développées par le système pour faire face à ces perturbations et les effets qui en résultent. Nous nous appuierons pour cela essentiellement sur les données issues de la seconde année de recherche.

II - PRESENTATION DU LOGICIEL EUCLIDE

Le logiciel Euclide est un langage dérivé de LOGO. Conçu par des enseignants de mathématiques animateurs à l'IREM de Grenoble, il se veut une extension de LOGO spécifiquement adaptée à l'enseignement secondaire de la géométrie [1].
Il inclut pour cela un certain nombre de macro-procédures permettant d'accéder directement aux objets et notions manipulées en géométrie à ce niveau (en France), ces procédures étant gérées par une syntaxe proche de la syntaxe mathématique usuelle :
- Il existe ainsi des macro procédures correspondant aux définitions usuelles des objets classiques : droites, cercles... Une droite par exemple peut être définie à partir de 2 points A et B par l'instruction DRPP :A :B, comme droite parallèle (resp. orthogonale) à une droite donnée D passant par un point A par l'instruction DRPAR :D :A (resp. DRORT : D :A), comme médiatrice d'un segment par l'instruction MEDTR :A :B, comme bissectrice....
- Il existe aussi des macro-procédures pour toutes les transformations usuelles et on définit par exemple le point B comme image du point A dans la rotation de centre C et d'angle 30° par l'instruction : SOIT "B ROT :C 30 :A.
- Il existe enfin des fonctions de mesure pour les longueurs et les aires.

Une caractéristique qui se révèle didactiquement importante est la suivante : les objets peuvent être définis et ceci se fait par la succession : SOIT "Nom de l'objet Instruction. Ils peuvent être également tracés sur écran sans définition préalable par la succession : DES Instruction. Ainsi :
SOIT "C1 CLDM : A :B définit le cercle C1 de diamètre [AB] mais ne le trace pas,
DES :C1 trace le cercle C1 précédemment défini,
DES DRPP :A :B trace la droite (AB), les points A et B ayant été préalablement définis, mais ne la définit pas.
Cette distinction permet d'alléger les tracés mais est source, on le montrera, de difficultés résistantes, par les disparités qu'elle introduit avec le fonctionnement mathématique usuel.

III - PRESENTATION GLOBALE DE LA RECHERCHE

Cette recherche s'est engagée à l'IREM Paris 7 en octobre 1988 dans le cadre d'un projet national inter-IREM. Elle a concerné trois classes de quatrième pour la phase exploratoire de la première année, deux classes la seconde année et une la troisième. Le travail de la phase exploratoire a donné lieu à un premier rapport intermédiaire [2], celui mené la deuxième année est en cours de publication.

III-1 : L'étude des potentialités du langage Euclide

Cette étude a pris en compte deux directions :
- l'exploitation via la réalisation d'imagiciels par l'enseignant,
- l'exploitation via la programmation par les élèves eux-mêmes.

Nous présentons ci-après les hyptohèses qui ont fondé le travail dans la deuxième direction, celle qui nous intéresse ici. Ces hypothèses n'ont rien d'original. On les retrouve, plus ou moins explicitées, dans beaucoup de recherches concernant l'enseignement de la géométrie dans un environnement informatique, tel ou tel aspect étant plus ou moins souligné suivant les caractérisitques du suppport informatique considéré et le niveau d'enseignement envisagé [3], [4].

- Euclide peut aider la formulation en géométrie. En effet, comme tout langage informatique, il impose des exigences de rigueur dans le domaine syntaxique. On fait l'hypothèse que l'adaptation à ces exigences, du fait de la proximité du langage Euclide du langage mathématique usuel, va produire des connaissances transférables au fonctionnement mathématique usuel, tout en ne nécessitant pas d'apprentissages annexes trop coûteux.
On fait aussi l'hypothèse que le recours à Euclide présente l'avantage didactique de mettre en scène ces exigences comme des exigences du "milieu" et non comme des exigences relevant du simple "contrat didactique", comme c'est trop souvent le cas (cf. paragraphe IV).
L'intérêt de ces hypothèses est justifié par le fait que les problèmes de formulation mathématique, tels qu'Euclide permet de les prendre en compte, sont loin d'être des problèmes résolus en classe de quatrième et qu'ils y constituent encore un enjeu de l'enseignement.

- Euclide peut aider la conceptualisation en géométrie. En effet, d'une part il est bien connu que la conceptualisation n'est pas indépendante de la formulation (cf. [5] par exemple), d'autre part, Euclide oblige à envisager différemment les objets géométriques. Il possède à ce niveau des caractéristiques relativement proches de celles du logiciel "Cabri-Géomèire" telles que décrites par exemple dans [3]. Dans la géométrie usuelle du collège, la perception, les instruments perçus comme objets permettant d'exécuter des "gestes" jouent un rôle dominant, quels que soient les efforts faits par l'enseignement pour contrecarrer cette tendance [6]. Dans la géométrie d'Euclide, ces gestes doivent être décomposés, analysés et traduits en termes d'objets géométriques et de leurs propiétés : on ne fait pas glisser une règle, on trace une parallèle, on ne pose pas une équerre mais on trace des droites perpendiculaires et, si l'on veut reporter des distances, il faut tracer des cercles ou utiliser des transformations géométriques.

- Euclide peut aussi aider à approcher dès ce niveau des situations plus complexes et permettre d'engager les élèves dans une démarche expérimentale en mathématiques, recherche de configurations respectant des conditions données, premiers problèmes de lieux, par exemple.

Euclide peut enfin aider les élèves à entrer dans la rationalité mathématique en les aidant à prendre conscience de la généralité des énoncés mathématiques : un programme associé à une configuration étant donné sous forme de procédure, il n'y a aucune difficulté a priori à tracer un grand nombre de figures associées à cette configuration et faire apparaître les propriétés géométriques de la configuration pour ce qu'elles sont, à savoir les invariants d'une large classe de figures (théoriquement infinie).

III-2 : Les directions privilégiées dans l'analyse du fonctionnement du système didactique

Ces directions, contrairement aux hypothèses précédemment citées, n'ont pas été choisies a priori. Elles se sont en fait imposées peu à peu au cours de la phase exploratoire de la première année du fait des décalages observés entre les hypothèses faites et certains résultats obtenus, des difficultés rencontrées à faire vivre certaines séquences ou certains moments de l'enseignement. Ce sont les suivantes :

- environnement informatique et milieu,
- environnement informatique et activité de l'élève,
- environnement informatique et pilotage de la classe par l'enseignant.

III-3 : Méthodologie

La recherche a débuté comme une recherche d'ingénierie didactique classique [7] : élaboration et analyse a priori de séquences s'appuyant à la fois sur la théorie des situations didactiques et sur les hypothèses spécifiques à la recherche, expérimentation, confrontation des données issues de la réalisation à l'analyse a priori pour tester les hypothèses à la base de la construction.
A partir de la deuxième année, elle a été complétée par des analyses longitudinales de type étude de cas permettant une étude plus fine des problèmes relevant du milieu ou des niveaux d'activité des élèves : suivi de deux groupes de trois élèves, un groupe de garçons et un groupe de filles la deuxième année, un groupe mixte de deux élèves cette année.
Cette année enfin, les séances collectives de synthèse qui font suite systématiquement aux séances de travail en groupe devant les ordinateurs sont systématiquement enregistrées. Confrontées aux préparations faites par l'enseignant pour ces synthèses, elles nous serviront à étudier ses prises de décision dans le pilotage de la classe.

IV - ENVIRONNEMENT INFORMATIQUE ET MILIEU

IV-1 : La notion de milieu

L'expression "milieu" est utilisée dans ce paragraphe avec l'acception qui lui a été donnée par G.Brousseau [8]. Cette notion est essentielle dans une perspective constructiviste de l'apprentissage car, comme l'écrit G.Brousseau dans le texte cité :
"L'élève apprend en s'adaptant à un milieu, qui est facteur de contradictions, de difficultés, de déséquilibres, un peu comme le fait la société humaine. Ce savoir, fruit de l'adaptation de l'élève, se manifeste par des réponses nouvelles qui sont la preuve de l'apprentissage."
Dans une telle approche, l'enseignant n'a pas pour charge de transmettre directement des savoirs mais d'aménager, de mettre en scène le milieu de manière à ce que les relations de l'élève avec ce milieu produisent des adaptations correspondant aux apprentissages souhaités.De plus, cette mise en scène, tout en étant le fruit du travail didactique de

l'enseignant, doit permettre de présenter à l'élève un milieu lavé de toutes ces intentions didactiques pré-existantes, a-didactique selon la terminologie de G.Brousseau : la réponse de l'élève ne doit pas être motivée par les obligations du contrat didactique, l'anticipation du désir du maître, mais par des nécessités a-didactiques de relation avec le milieu.

Les problèmes rencontrés dans cette recherche nous ont amenés à distinguer dans le milieu deux composantes :
- la composante inerte,
- la composante active, engagée dans l'apprentissage.

La première est constituée de ce qui dans le milieu est supposé bien connu, transparent pour l'élève, soit du fait de la culture ambiante, soit du fait d'apprentissages antérieurs et c'est l'adaptation à la seconde qui doit provoquer l'apprentissage.
Ce schéma fonctionne efficacement dans les environnements mathématiques usuels d'apprentissage qui ne font intervenir généralement que des milieux assez pauvres : la composante inerte n'y parasite pas trop le fonctionnement de la composante active.
La situation n'est plus la même dans le cas d'un environnement informatique : ce que l'enseignement voudrait pouvoir supposer inerte dans le fonctionnement de la situation didactique ne l'est pas nécessairement et l'on peut faire l'hypothèse que le système, peu habitué à fonctionner sous de telles contraintes, va avoir des difficultés à s'y adapter.
Dans ce paragraphe, c'est à l'étude de ces phénomènes que nous allons nous attacher.

IV-2 : Caractéristiques et fonctionnement du milieu dans la recherche

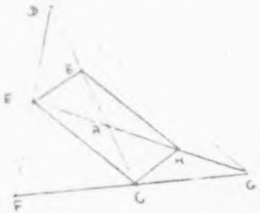
Dès la première année, l'expérimentation a essayé de prendre en charge les problèmes de milieu, par un certain nombre de choix didactiques globaux, en particulier :
- organisation d'une familiarisation avec l'environnement informatique en préalable à l'utilisation du logiciel Euclide sous forme d'un module LOGO de plusieurs séances visant à assurer pour tous les élèves la transparence de différents aspects du milieu : l'objet ordinateur, la manipulation du clavier, les entrées et sorties, la notion de langage, le mode direct et le mode procédural,
- association à chacune des séances de travail en petits groupes en salle informatique d'une séance de bilan et institutionnalisation, gérée collectivement et prenant en compte les deux dimensions : mathématique et informatique,
- constitution d'aides-mémoire informatiques.

Ils se sont révélés insuffisants et les résultats de l'évaluation finale organisée en environnement informatique ont en particulier mis en évidence de nombreux phénomènes de ralentissement voire de blocage liés à de tels problèmes, non mathématiques [2].

Le dispositif mis en place la seconde année a permis d'analyser ces phénomènes de façon plus fine et nous a amenés à classer les apprentissages nécessaires des élèves dans leur relation au milieu et les erreurs commises selon une échelle d'imbrication entre mathématiques et informatique.

L'apprentissage de la syntaxe dans ses aspects imbriqués aux mathématiques est une réponse attendue aux réactions de la partie active du milieu : l'adaptation à ces contraintes syntaxiques est conçue en effet comme moteur de l'apprentissage mathématique (cf. hypothèses). Mais malheureusement, les apprentissages nécessaires sont loin de se réduire à cet aspect et, comme tendent à le montrer les résultats obtenus, c'est justement aux niveaux les moins imbriqués que se situent surtout les erreurs recurrentes. En particulier, les données recueillies montrent que ce type d'erreurs réapparaît dans les situations complexes où l'attention n'est plus focalisée, comme c'était le cas dans la phase d'initiation, sur l'apprentissage du langage (cf. en français orthographe dictée/orthographe rédaction), qu'il resurgit aussi après toute interruption un peu longue dans l'utilisation d'Euclide (vacances par exemple), ralentissant le fonctionnement des groupes ou créant même des blocages, et montrant bien par là-même que cette composante du milieu n'est pas aussi inerte qu'on le souhaiterait.

Dans ce texte, nous illustrerons les affirmations qui précèdent par les données issues d'une tâche de construction proposée aux élèves à l'issue de la phase d'initiation au langage Euclide. Dans la présentation orale, nous complèterons ces données par celles de protocoles de travail de groupes en cours d'année ou de travail individuel pendant l'évaluation finale.
En ce qui concerne la tâche de construction, dans un premier temps les élèves devaient proposer des constructions partant de 3 points librement choisis pour la figure ci-après :



Chaque élève devait produire à la fois un texte mathématique et un programme Euclide. Nous donnons ci-après le nombre d'élèves ayant construit correctement au moins n points, d'abord du point de vue mathématique et ensuite du point de vue Euclide :

≥ n	8	7	6	5	4	3
Maths 7	8	9	13	17	21	21
Euclide	3	3	6	10	11	17

Ce tableau met bien en évidence, nous semble-t-il, le décalage existant entre la correction mathématique et la correction Euclide. Pour mieux cerner cette différence, nous allons analyser les erreurs commises dans la tâche qui a suivi celle-ci immédiatement dans l'enseignement : le professeur, ayant repris les cinq types de construction trouvés par les élèves, a demandé de coder en Euclide cinq textes mathématiques correspondant à ces constructions.

Nous distinguerons pour les besoins de cette analyse 5 catégories d'erreurs, par ordre de désimbrication mathématique croissante:

- *Les erreurs mathématiques :* non définition de certains objets ou double définition, ajout de propriétés (code : **EM**).
- *Les erreurs correspondant à un mode de définition mathématique non directement traduisible en Euclide* (code : **ED**), ex : on peut en mathématiques définir G tel que C soit le milieu de [GF], en Euclide, une instruction définissant G ne peut faire appel à G, donc il faudra par exemple définir G comme le symétrique de F par rapport à C, F et C étant bien sûr préalablement définis.
- *Les erreurs correspondant à un changement d'ordre entre la syntaxe Euclide et le langage mathématique usuel* (code : **EO**), ex : "M' est le symétrique de M par rapport à O" s'exprimera en Euclide par :

SOIT "M' SYMP :O :M

- *Les erreurs liées à une définition implicite en mathématiques mais nécessaire en Euclide* (code : **EI**), ex : en mathématiques, dès que deux points A et B ont été introduits, on peut parler de la droite (AB), en Euclide, cette droite doit être à son tour définie ; il ne suffit pas non plus qu'elle soit tracée pour être définie.
- *Les erreurs dites de syntaxe "de liaison"* (code : **EL**) : espaces, : , " , [....

Le tableau ci-après donne pour chaque type d'erreur le nombre de fois où elle a été commise dans le décodage des 5 textes et le nombre d'élèves concernés sur un effectif de 21 :

Code erreur	EM	ED	BO	EI	EL
Nombre	4	7	32	56	28
Elèves	1	2	8	18	9

Note : dans le type EI, il s'agit uniquement de l'utilisation de droites non définies, mais dont deux points ont été préalablement définis.
Ce tableau montre bien la prédominance dans les erreurs, à ce moment de l'enseignement, des types correspondant aux niveaux de moindre imbrication.

V - INTERPRETATION ET CONCLUSION

Les résultats obtenus dans cette recherche Euclide mettent en évidence deux grandes catégories de difficultés rencontrées par les élèves dans leur adaptation à l'environnement informatique considéré : celles présentant une imbrication mathématique forte et à l'inverse celles présentant une imbrication mathématique faible, voire nulle. Ces deux types de difficultés ne vivent pas de la même façon dans le système didactique : les premières sont naturellement enjeu d'enseignement donc objet d'attention. Les données obtenues montrent qu'elles tendent à disparaître au cours du processus d'enseignement, montrant par là-même l'impact du travail Euclide sur l'apprentissage mathématique. Les secondes résistent et réapparaissent alors même que l'on croît avoir réussi à les éradiquer par la phase de familiiarisation. En fait, on peut faire l'hypothèse que les connaissances auxquelles elles correspondent sont vécues par les différents protagonistes, élèves et enseignants, comme des connaissances hors-contrat en mathématiques, ne pouvant donc faire l'objet d'évaluation ou de sanctions, de façon continue. Cette hypothèse semble se trouve confirmée par l'analyse a posteriori du traitement fait des erreurs syntaxiques dans les évaluations en dehors de la phase d'initiation : les erreurs désimbriquées y sont en effet traitées, à l'inverse des erreurs imbriquées, comme des erreurs vénielles. Ceci pose un réel problème didactique dans la mesure où, la recherche menée le montre clairement, l'efficacité de l'utilisation d'Euclide avec programmation par les élèves, passe par un apprentissage du milieu sous ses différents aspects, imbriqués aux mathématiques ou non, et que, semble-t-il, l'on ne peut assurer pour l'instant, par la seule culture ambiante ou de brèves phases de familirarisation, l'inertie souhaitable de ce milieu informatique.

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PROCEDURAL AND RELATIONAL ASPECTS OF ALGEBRAIC THINKING

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Summary. *The paper discusses some results of empirical research made with pupils from 10 to 16 years, while solving pre-algebraic and algebraic problems. The starting point is the analysis of epistemological obstacles to algebraic thinking; afterwards, the concept of double polarity, i.e. procedural vs. relational, is introduced in order to describe processes and strategies of pupils. Algebraic thinking is deeply characterized by double polarity: when both aspects are present and active, pupils can master real algebraic strategies, while when one is missing, pupils' strategies reveal typical faults, already described in the literature. To support the points, some examples are briefly sketched.*

Introduction. Many researches of these last years have shown the formidable importance of **epistemological obstacles** in learning algebra (see Brousseau [83], Chevallard [85], Laborde [82], Margolinas [89], Vergnaud [89]): e.g., arithmetic thinking seems to be a typical obstacle to the developing of algebraic thinking (see: Kieran [89]). In teaching algebra to pupils from 11 to 16 (and over), it is not very easy to plan **learning situations** with a good **validation** (see Margolinas [89] for a discussion of boldfaced terms). The point is that an effective construction of algebraic knowledge as a net of operative ideas and algorithms is difficult: in fact, it is easier to produce a crude memorization of mechanical rules than a thorough use of them (see the convincing examples in Kieran [89]). From another point of view, some researches (see Vergnaud [89]) have emphasized the paramount importance of so called pre-algebraic thinking in very young pupils: e.g.,

when children use arrow-diagrams to solve inverse problems in elementary schools, they do reason in an algebraic fashion, according to Vergnaud. Many other examples are supplied by pupils while using a computer (cf. the joint paper by G.P. Chiappini and E.Lemut in this volume).

The research discussed in the paper, which is still developing, studies the apparent paradox between an approach to algebra, which seems possible at an early stage from one side, and the difficulties of getting good results at a more advanced level of age, from the other. It is based upon the observation of more than 150 pupils from 9 to 16 years, faced with pre-algebraic and algebraic problems (a definition of pre-algebraic will be given at the end of the paper: see §3c). The results support evidence for the following thesis, which overcomes the apparent contradiction between the two points of view: "**A double polarity**, procedural vs. relational, characterizes algebraic thinking and allows pupils to achieve algebra in a constructive fashion. A natural way to develop this double polarity is to approach algebra by means of so called **concrete mathematics** and to use it as the natural support for generalization, manipulation of formal objects (i.e. formulae), abstraction". The term concrete mathematics is here used in a very precise meaning, namely that discussed in Graham et al. [89]: "Concrete Mathematics is a blend of CONtinouous and disCRETE mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using a collection of techniques for solving problems"(p.V). The problems in the *Appendix* are examples of concrete mathematics.

The paper is divided into three parts: the first focuses the main obstacles in the learning of algebra; the second (and most important) discusses procedural and relational polarities in algebraic problem solving, illustrating them with some examples; the third sketches some problems related to the process of generalisation in algebra, from the point of view of the double polarity. An *Appendix* contains the summary of pre-algebraic and algebraic problems, quoted in the paper.

1. Obstacles. It is very well known that many pupils, from middle school to the university, solve algebraic problems more in a 'syncopate' style than in a 'symbolic' one (see Harper [87] or Kieran [89]). Pupils of 16 years do not jet use the algebraic code spontaneously while solving simple algebraic questions (e.g. Problem 1 in *Appendix*), even if this may cause them troubles and long detours. An arithmetical style seems to prevail and this may cause conflicts with the algebraic way of thinking. The point is that algebra is understood only in abstract, as an abstract and general method, but it is not concretely used as a method of justification and generalisation; namely, generalisation as an effective and operative method is not used by pupils, while solving algebraic problems: they live more comfortably with its substitutes. Students generally learn at school how to solve specifical problems with algebra (e.g., using unknowns), but major difficulties are met when they are required to use it as a symbolism to express general solutions, to prove or verify laws behind numerical relations, etc..

As C.Laborde has pointed out in her thesis [82], the main functions of the algebraic code, namely *individualization* and *linkage*, are substituted with extralinguistic properties. The language of pupils faced with problems like those in the *Appendix*, is plenty of linkages to different aspects of the real space-time situation in which they act, while producing their own solution of the problem. In their protocols it is easy to find a melting pot, whose ingredients are both mathematical and extramathematical, extralinguistic, etc.. Typically, mathemathical objects are referred to by means of subject's actions (e.g., processes of

calculus made by the subject himself, or by somebody else), algebraic laws are put into the flowing stream of time. All this make easier for pupils to remember the meaning of formal things, they are speaking of, and to control the situation.

Pupils express the meaning of mathematical objects, relating them in some way to: subject's actions, the very processes of their constructions and generations, every other extramathematical information about them. This is a major root of the obstacles to symbolisation and to purely syntactic manipulation of symbols (particularly when condensed in literal blocks: see Norman [86]); in fact, this causes an *evaporation* of extra-mathematical data and, consequentely, a possible dramatic loss of meaning.

2. The double polarity. Every solution of an algebraic problem like those in the *Appendix* lives dialectically in a double polarity. From one side, there is the subject, who solves the problem, with his actions in the flowing of time: the algebraic code is interpreted essentially in a **procedural** manner. From the other side, the algebraic code is interpreted in an absolute way, independently from the actions of anybody; it is a contemplation of relations and laws *sub specie aeternitatis*; neither procedures nor products of actions are involved: in fact only the **abstract-relational** aspect remains and its privileged code is symbolic. For ex., pupils who attack problem 2, first mimic concretely messenger's and prince's trips, and score in some way the exact time when they meet each other; after the third meeting, concrete manipulations become too difficult, so they try to foresee "the rule" and check it (procedural polarity). When requested to justify the general rule, they have found, things are viewed from a relational point of view, and their reasoning can be summed up as follows (I use letters, like d, S, etc. for conciseness'sake; children like better other detours, e.g. the use of concrete numbers as variables): "suppose the messenger and the prince separate after d days, say in point S; it takes d days more to the messenger for reaching the castle and coming back to S; in the meanwhile the prince arrives at a point I; in the end, exactly in d days the messenger reaches him at a point E, symmetric of S with respect to I". The transition from procedural to relational polarity is even more marked for strategies of solutions in problems 3, 4.

The main features of the double polarity with respect to the birth and development of algebraic thinking can be sketched as follows.

Procedural polarity is a-simmetric, has a privileged direction, is controlled by means of tense adverbs and prepositions; its logic is the "logic of when". Relational polarity is ruled by logical and equivalence laws, which are typically symmetric: its logic is the "logic of if and only if". The former allows a strict semantic control: extra-linguistic facts link steadly the speech to subject's actions in the flowing stream of time. The latter, on the contrary, is typically syntactic: concrete meaning has *evaporated* and only symbolic objects have been preserved. They have their own semantic, but it is a very formal one, as depends upon symbolic code and no longer on previous extra-linguistic facts. To be more precise, in problems 3 and 4 pupils first work in procedural polarity and find the recurrence law, which expresses locally the solutions (for ex. $S(2n) = 2S(n)-1$, in problem 3); afterwards, they manipulate it more formally, almost forgetting the original meaning of the symbols, in order to eliminate the parameter n (i.e. time) from the formula and to get the global solution .

In the procedural polarity, the dominating **epistemological style** is **arithmetical**. Calculations are performed as soon as possible: every (syntactical) term is developed at once, until it is reduced to an irreducible one (number written in a canonical form). The epistemological model compels to reduce formal complexity of terms, without caring of their numerical complexity, namely the number of digits required to write them, e.g. in base 10. On the contrary, the epistemological style of algebra sometimes requires to increase the formal complexity of a term, at least locally, possibly preventing the growth of its numerical complexity: e.g., to solve problem 1 one must extract the identity $(n+1)(n-1) = n^2-1$ from numerical examples, like $7 \times 9 = 8 \times 8 - 1$, i.e. one must not only stop calculations but also increase the formal complexity of terms 7, 9, writing them as (8-1), (8+1). That is, one must rule things against arithmetical style. Analogous observations are true for the manipulations needed to pass from (recurrence) local formulas to (explicit) global ones in problems 2,3,4.

The most difficult point is that such manipulations become active if (and generally only if) are approached from procedural polarity, but can be done effectively only destroying extra-mathematical tracks, that is eliminating this very polarity, which is the semantic base of syncopate algebra. Sometimes, a main product of this process is what I call **condensation**, a typical phaenomenon of algebraic thinking. It means that using the symbolic code, one can write concisely and expressively the amount of information of a term, whose complexity cannot be easily ruled with the syncopate language of arithmetic. Condensation is not the result of a process as much, but rather a general attitude with respect to the very process (what I call a procedure). Condensation marks deeply the passing from a procedural moment to a more abstract and relational one; it appears at once in the strategies of solutions of pupils and, as all processes which happen on the spot is very difficult to analyze. For ex., in all examined cases, most pupils who worked successfully at our problems could solve them, passing from procedural to relational polarity, with a sudden change of their style. However, even when explicitly asked, they were not able to explain what had happened.

3. Abstraction?. To conclude, some general remarks and some sentences which jet need to be supported by empirical evidence.

a. The discovery-construction of an algebraic rule is not a trivial process of generalization from particular to general, but it is stirred by the strained connections between the two polarities. Typically, the dialectic between the two polarities marks the birth of algebraic work.

b. Relations between the finalities of algebra to answer a question and the very algebraic work have been identified (see Margolinas [89]) as the roots of **validation** (of the product) and **control** (of the procedure). In fact, finalities live in a procedural dimension, while algebraic work develops in a relational one: control and validation sometimes involve both, sometimes there is a form of **weak validation**, where only the second polarity seems to be involved.

c. At this point, it is possible to give the promised definition of pre-algebra: it is every activity which involves the first polarity in an effective way, while the second is involved in a more problematic way (and it is so not only because arguments are so, but also because of the 'didactic contract'). In other words, in pre-algebra procedural aspect is in tension and not a quiet way of life.

Appendix (Summary of problems quoted in the paper: in brackets information on the age of pupils, to whom the problem has been posed).

1. Take three consecutive numbers, calculate the square of the middle one, subtract from it the product of the other two; what is the result? now change numbers, and try again....Explain if and why the result is always the same [13 -16 years].

2. A prince decided to make a trip along his land and started with his followers. In one day they traveled 50 Km. Next morning, the prince sent back a messenger to his castle, while he continued his trip. The prince went on travelling 50 km every day, while the nessenger rode 100 km each day. How long a time before the messenger reached his prince? And if the history goes on, with the messenger who rides back and forth from the prince to the castle 100 km a day, while the prince goes on 50 km every day, how long does it take to meet the second, the third, the n-th time? [10-12 years].

3. There are n people at a round table, and we eliminate every second remaining person until only one remains. For ex., for n=10, the elimination order (starting to count from n.1) is: 2, 4, 6, 8, 10, 3, 7, 1, 9 and 5 survives. Determine the survivor's number, S(n). [14-16].

4. With a single cut, a big pizza can be divided into 2 parts; with 2 cuts (suppose to do straight cuts), a pizza can be divided at most into 4 parts; with 3 cuts one gets at most 7 parts, etc.. Which is the max. number of parts one can get with n cuts?[14-16 years].

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HEGEMONY IN THE MATHEMATICS CURRICULA: THE EFFECT OF GENDER AND SOCIAL CLASS ON THE ORGANISATION OF MATHEMATICS TEACHING FOR YEAR 9 STUDENTS

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This paper reports findings from part of a long term project into the social context of mathematis education in schools in Brisbane, Australia. Four schools from different socio-economic backgrounds participated in the stuydy. One of the aims of the project was to study how hegemony operated in the schools with teachers and students. This paper discusses factors related to the prescribed syllabus, the dis-empowered teacher and the student background.

Since the mid sixties, the mounting evidence of the failure of many school compensatory programs to raise the conditions of minorities and the disadvantaged has put under question the role that schools play in reproducing social and cultural inequalities. Bowles and Grantis (1976) argued that schools play a primary role in preparing students from different social backgrounds to meet the needs of an unequal society. This position has been widely supported in the literature (Apple, 1981; Giroux, 1983; Willis, 1977), however Bowles and Gintis' (1976) theory of 'correspondence' between the power relations in the school and the workforce has been criticised for presenting a rather deterministic view of the role of schools in reproducing inequalities in society. Hegemony (Gramsci, 1971) is now seen as the process by which the dominant classes of society use ideology, if not brute force, to impose their views and values on the needs and interests of the subordinate groups.

Mathematics appears to have a formidable role in reproducing crucial ideologies, a role made more powerful because of the notion of objectivity that surrounds mathematics itself that masks its hegemonic function. Bishop (1988) argued that mathematics is neither culture-free nor value-free. There is evidence that teachers see mathematics as crucial subject for reproducing existing social values, and that it functions as a "behavioural badge for eligibility for the various privileges of society" and that they modify their curriculum accordingly (Stake and Easely, 1978).

An Australian national review of teacher education in science and mathematics has agreed that a statement summarising the status of mathematics teaching in the United States approximates the situation in Australia:

> We have inherited a mathematics curriculum conforming to the past, blind to the future and bound by tradition of minimum expectation (Department of Employment, Education, and Training, 1989).

Bishop (1988) presented a similar picture of a mathematical curriculum dominated by techniques, driven by learning conceived independent of the learner, bound by textbooks, and dominated by the needs of tertiary mathematics.

This paper discusses various forms in which hegemony operates in the school. The discussion is based on observations obtained from the second stage of a project on the social context of mathematics teaching in high schools carried out at the Centre for Mathematics and Science Education, a project whose long term aim is to produce guide-lines for appropriate curriculum for the needs of students from lower socio-economic classes and girls. In particular, this paper discusses hegemonic factors that limit teachers and schools from adopting mathematics curriculum that empowers all students to increase their achievement in, and enjoyment of, school mathematics experiences.

THE STUDY

This second stage of the project was a field study (Popkewitz and Tabachnick, 1981) in four state high schools in the metropolitan area of Brisbane. The schools were selected to represent different socio-economic backgrounds: professional, middle class, working class and welfare. All four schools were well known as examples of their type and were innovative in their programs and curicula.

There were some similarities in the mathematics curicula and teaching in the four schools. They all followed the same official syllabus, their teachers followed the same traditional 'chalk and talk' approach to teaching and their learning activities were dominated by textbooks and by the teacher. Each school allowed some variations in the type of mathematics programs available, however for all schools this variation was limited, under-utilised and inadequate for innovative mathematics teaching.

A single grade 9 (second year in the secondary schools in Queensland) was selected from each school and a multiple instrument technique was employed to collect information about the mathematics teaching in these classes. Each class was observed for the duration of one topic (between one and two weeks). A classroom observation instrument, modified from Good and Brophy (1987) by the authors in the first stage of the project, was used to record and classify teacher-student interactions. Each lesson was audio-taped and the tapes transcribed. In addition, interviews were conducted with the principal and mathematics subject master of each school, each class teacher and selected students from each class, to provide a comprehensive view of the different aspects of the mathematics curriculum in the schools and to allow for triangulation with the observations. Teachers' and students' perceptions of mathematics, their own and each others' abilities and attitudes, and each others' perceptions, were the primary focus of the interviews with the students and the class teachers (see Cooper and Atweh, 1991). The interviews with the principals and subject masters (and parts of the interviews with the class teachers) focussed on the culture of the school and the role of mathematics and provide the basis of this paper.

HEGEMONIC FACTORS

The Prescribed Syllabus

In all schools, the textbook or work program used in each school closely followed the state Education Department syllabus. This was seen by all principals, subject masters and class teachers as the major factor in determining classroom mathematics teaching. In Queensland, officially authorised subjects, called Board Subjects, such as mathematics tend to be of an academic nature and their major purpose tends to be contributing to students' Tertiary Entrance scores. In spite of a general acknowledgement that the school population is changing, and that this requires schools to diversify their curriculum, the principals, subject masters and class teachers retained a conception that the major function of schools is to prepare students either for university studies or skilled occupation. Mathematics was perceived as an important subject for both purposes. The distinction between high prestige academic and low prestige practical subjects was maintained in all schools. This was a cause of concern in all schools but more so in the welfare class school. Very few of their students attained achievement levels that would allow them the option of university study. The majority became unemployed or found unskilled work.

Diversifying the schools' offerings of non-board subjects in mathematics to suit the varied needs of the student population raised different dilemmas in each school. The principal of the welfare class school was quite aware of the two sides of the debate. On one hand, he could see that students in the non-academic streams had significantly improved self esteem. Yet, he was quite concerned by the possibility that such options would limit students' chances for advanced study in mathematics required for many professions and careers.

Dis-empowered Teachers

There have been some major changes in the official Queensland mathematics syllabus. A recent P-10 innovation in the syllabus has encouraged greater use of activity learning and a focus on mathematical processes rather than facts and skills. However, this innovation has appeared from the interviews to have contributed to a feeling of teacher helplessness. The subject masters and the class teachers stated that they have not been given adequate in-service training to be able to successfully put the new curriculum into the practice of their classrooms. The subject master from the working class school said that teachers felt threatened by the new syllabus. The subject masters and the teachers did not appreciate an implication in the syllabus notes, where changes were justified as necessary to improve the teaching of mathematics, that their existing teaching was inadequate. Secondly, since the new guide-lines de-emphasised the use of textbooks, they were concerned that no extra time was given for teachers to prepare work programs and activities. Lastly, they feared to venture into innovative programs in case the common testing arrangements, evident in all schools, would identify their students as being behind other students in skill on exercises. Such insecurity gave the teachers a feeling of dis-empowerment and de-skilling.

This feeling of dis-empowerment was evident from another source. In Queensland, it is a common practice to have teachers teaching mathematics in the secondary school with little or no tertiary mathematics or mathematics education study (such teachers are more common in lower socio-economic and country schools). Some of these teachers end up teaching mathematics because their subjects are no longer offered. Others are transformed from the primary school, or from other discipline areas, with minimal in-service training. Three of the four class teachers participating in this study fell into this category. Two of these teachers, from the working class and welfare schools, were highly dedicated; yet had a sense of insecurity which made them wary of varying their teaching from the traditional 'stand and deliver' or 'chalk and talk' approach. However, the third relatively untrained mathematics teacher, at the professional school, provided an innovative program based on an appreciation of the nature of mathematics as a language for communication. This was based on the teacher's interest in language and primary teaching skills.

The dis-empowerment of teachers appeared to lead to dis-empowered teaching. To cope with the insecurity, the teachers were happy to follow someone else's interpretation of the guidelines in the form of a work-program of a textbook. This appeared to lead to an image of mathematics in the minds of teachers, and students, characterised by rigidity, rules and facts and recall. The result of this was an observed classroom behaviour that can best be classified as dis-empowered learning, a failure to see value in learning mathematics or to enjoy its study. The teachers in this study expressed the opinion that a major role for teachers is to find a way to externally motivate students. The teachers at the working class and welfare schools expressed concern about their lack of knowledge of how to teach mathematics. There appeared to be a chain reaction of dis-empowerment, a cycle in

which the dis-empowered learner adds to the sense of guilt that the teachers have about their own limitations.

Student's Background

Unfortunately, the interviews appeared to show that student characteristics such as gender and social class still determine the opportunity to learn mathematics. The qualifications and experience of teachers in the different schools visited, as described by the subject masters, was related to the social class of the school.

In Queensland, funding to a school is mainly determined by the number of students in the school. Although, some underprivileged schools receive some additional assistance, these funds are hardly sufficient to enable the school to deal with its many needs. Wealthier schools can impose high levies on their students and receive more support from their Parents and Citizens organisations.

In the working class and welfare schools, the teachers appear to be ignorant of the culture of the students. One subject master confessed that it was difficult for him to design a program to meet the needs of his students because he was not from the same social background. There was a tendency in the interviews for the teachers and the subject masters to lay the blame for poor achievement, even when acknowledging its social class basis, at the door of the individual students - "if they want to learn, they can learn". However, this was not black and white as the following quotation shows:

> Their lack of experience makes it difficult for them to cope with any subject. They do not bring to the subject much background. They do not bring vocabulary. They do not bring experiences. They do not bring earlier concepts. I do not think it is just the primary school; you can't blame them. I think it is the home.

In another part of the interview, this teacher acknowledged that these kids do not suffer from lack of intelligence; "They are street-wise. They have animal cunning." What the teacher appeared to be expressing was all too common in the interviews: a failure to come with grips with the fundamental question of how to make mathematics experiences relevant to the background of the students.

Similarly, gender appeared in the interviews as a factor in learning mathematics. This was most obvious in the professional school visited. This school saw itself as being in competition with the elite private schools of Brisbane. There were more girls than boys at this school, especially at the higher years. This was seen by the principal as part of a tendency within many families in the neighbourhood to send their sons to a prestigious boy's private school while they send their daughters to the local state school. The hidden message given to these girls is not difficult to imagine. Likewise, all interviewees felt that there was more pressure put on boys than girls from parents to attempt the highest levels of mathematics. The class teacher in the professional school was very aware of this problem and the disadvantage it brought to the girls. He made a point of commending girls at every opportunity. However, there was no recognition of the special needs of girls in the working class and welfare schools. Teachers denied that they needed to deal with girls and boys any differently. There were no special programs encouraging participation by girls in mathematics in these schools.

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INSTANTANEOUS SPEED : CONCEPT IMAGES AT COLLEGE STUDENTS LEVEL AND ITS EVOLUTION IN A LEARNING EXPERIENCE[1]

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The report I present here is part of a research about the learning processes of derivatives and about the evolution of the concept image of 15-16 years old pupils in three classes. Students are asked about instantaneous speed as an exemple of derivative. From the analysis of the results we have obtained three pupils **profiles**, the "Primitive" one, the "Approximation" one and the "Limit" one, which characterise their concept images at each moment and which have revealed to be a good instrument to describe them and to observe their evolution along a learning experience. Here we explain the main features of these profiles and the evolution we have observed.

1- Présentation de la recherche

La recherche que nous présentons fait partie d'une étude sur l'évolution du processus d'apprentissage de la notion de dérivée. Elle a éte faite dans trois classes de "Segundo de BUP" (15-16 ans) de la zone géographique de Barcelone, dont les professeurs de Mathématiques enseignaient le sujet "Derivées" avec le matériel curriculaire *Introducció a las derivadas* (Grup Zero,1987).

Les principales caractéristiques du matériel de classe *Introducció a les derivades* peuvent se résumer comme suit :

- cet étude précède l'étude de la notion de limite, c'est à dire qu'on fait un traitement non rigoureux du concept de limite.
- on emploie une méthode géométrique qui consiste à identifier la derivée d'une fonction en un point avec la pente de la droite tangente au même point.
- on emploie une méthode numérique qui consiste à calculer la pente de la droite tangente par approximations successives à partir du calcul des pentes des sécantes (en employant la calculatrice).
- on présente la variation instantanée d'une fonction en un point avec l'exemple concret de la vitesse qui, d'un point de vue sémantique est l'exemple le plus facile de dérivée. Le mot "vitesse" accompagné de l'adjetif "instantanée" renvoie l'étudiant à ce qu'indique le vélocimetre : il a une existence réelle et il peut se mesurer.

Le recueil des données s'est fait en cinc temps :

1- Un premier questionnaire écrit, deux mois avant d'initier la phase d'apprentissage.
2- Une première entrevue, un mois avant d'initier la phase d'apprentissage.
3- Une deuxième entrevue, pendant la phase d'apprentissage et juste avant la leçon de formalisation de la derivée.
4- Une troisième entrevue, juste après la phase d'apprentissage.
5- Un deuxième questionnaire écrit, dix jours après la fin de la phase d'apprentissage.

Nous avons fait passer les questionnaires écrits à tous les élèves des trois classes (111) et nous avons fait les trois séries d'entrevues avec six élèves par classe, toujours les mêmes.

2- Premiers résultats à partir de l'interprétation graphique : les profils des élèves

L'une des phases de notre recherche a consisté en l'analyse des réponses des étudiants à quatre items observés au cours des entrevues qui portaient sur l'interprétation de graphiques de divers mouvements. Ces items sont les suivants:

Item A : "Determination de moments de vitesse maximum".
Item B : "Determination de la vitesse instantanée à un moment ou en un point donnés".
Item C : "Types de représentation graphique de la vitesse instantanée".
Item D : "Comparaison des vitesses instantanées de deux mobiles".

A partir des modèles de réponses obtenus à chacun de ces items, nous avons determiné trois profils d'élèves, que nous avons nommés "Primitif", "Approximation" et "Limite", et qui nous ont permis de caractériser le schéma conceptuel de chacun des étudiants à chacun des moments des trois entrevues, quant à la notion de vitesse instantanée[2]. Dans ce qui suit nous présenterons les principaux traits de chacun de ces profils.

a) Profil "Primitif"

Les étudiants qui présentent un profil "Primitif" manifestent une absence totale de schéma conceptuel de la notion de vitesse instantanée. Ces élèves répondent à des questions sur "la vitesse" sans percevoir la nuance des expressions "quand?", "où?", "à quel moment (point)?". Les élèves incorporent systématiquement les questions qu'on leur pose à leur schéma conceptuel de "vitesse", tout à fait vague, qui correspond à la formulation :

la vitesse est égale à l'espace divisé par le temps,

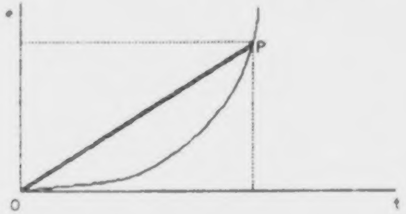
symbolisée par la formule

$$v = e/t.$$

On peut donc remarquer que ces élèves ne s'occupent pas du type de mouvement, ni du sens exact des mots vitesse, espace, temps ou de leurs symboles v, e, t, mais qu'ils se montrent bien convaincus de la correction de leurs réponses qui, dans d'autres contextes, ont certainement bien marché. Cela explique que ce schéma conceptuel de "vitesse" soit un obstacle pour affronter de nouvelles situations, où il s'agit de vitesses instantanées.

En ce qui concerne la représentation graphique de la vitesse en un point P ou à un moment donnés, nous avons obtenu deux modèles de réponse :

- La plupart de ces élèves se montrent surpris et déconcertés par la question qu'on leur pose et ne donnent aucune réponse.
- Quelques élèves (trois) tracent un segment qui unit l'origine des coordonnés (et du graphique du mouvement) avec le point P en question, comme sur le graphique suivant



Les procédés utilisés par ces élèves pour comparer deux vitesses sont basés sur un raisonnement qui est une espèce de corollaire de leur définition ; en effet, ils nous disent :

> si deux mobiles parcourent le même espace dans le même temps,
> ils ont la même vitesse.

Cependant, quand ils essaient de déterminer les moments de vitesse maximum, presque tous ces élèves se rapportent à des aspects locaux du graphique tout en utilisant des expressions telles que "à cause de la pente du graphique" (ou de l'"inclinaison" ou de la "forme" du graphique).

Nous voyons, donc, que le schéma conceptuel de la notion de "vitesse" des élèves de profil "Primitif", qui ne fait aucune distinction entre les notions de vitesse constante d'un mouvement uniforme, de vitesse moyenne d'un mouvement varié et de vitesse instantanée d'un mouvement quelconque, ne se montre pas toujours cohérent, bien que nous ayons pu apprécier sa grande stabilité.

b) Profil "Approximation"

Les étudiants que nous considérons ici, obtiennent la vitesse instantanée en un point donné, par approximation, ce qui consiste à calculer un taux moyen de variation entre deux points proches du dit point ; en d'autres mots, ils calculent la pente d'une droite sécante qui passe par deux points proches du point donné.

Ce qui caractérise ces élèves c'est qu'ils ont généralisé le schéma conceptuel de la notion de vitesse moyenne, qui leur servait autrefois pour décrire globalement un mouvement, à la notion de vitesse moyenne entre deux points proches, qui leur sert maintenant pour une description locale du mouvement, c'est à dire pour obtenir d'une manière approximative la vitesse en un point donné. Par conséquent, ces élèves répondent aux questions "quand?", "où?", "à quel moment (point)?" avec un schéma conceptuel de vitesse moyenne appliqué

localement.

Quant au calcul de la vitesse instantanée en un point P donné, le schéma conceptuel de ces élèves correspond à la formulation :

> la vitesse instantanée en un point P donné, s'obtient par approximation au moyen du quotient des incréments d'espace et de temps pris "proches du" point P,

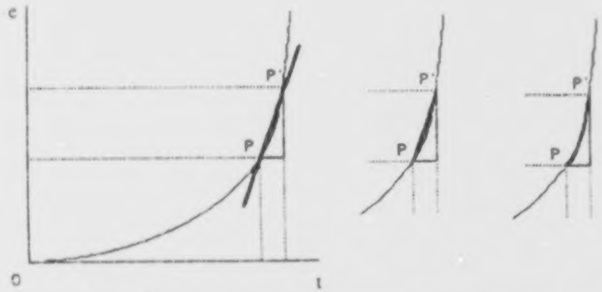
où l'expression "proches au" signifie parfois "autour du" et parfois "à partir du" ou "jusqu'au".

A partir de la seconde entrevue, les élèves de profil "Approximation" ont montré que leur schéma conceptuel comprend aussi une représentation symbolique qui prend plusieurs formes et que nous résumerons :

$$v(P) \approx v(PP') = \frac{e(t_2) - e(t_1)}{t_2 - t_1}$$

où v(P) est la vitesse instantanée au point P et v(PP') est la vitesse moyenne entre le point P et un point proche P'.

L'image graphique d'appui, qui correspond à ce schéma conceptuel, peut se représenter d'après les figures suivantes :



où les élèves tracent (ou s'imaginent) une droite sécante et les incréments correspondants de temps et d'espace, ou bien une corde qui forme un triangle rectangle avec les incréments, ou simplement un segment du graphique avec les incréments.

En ce qui concerne les items de comparaison de vitesses et de détermination de vitesses maximum, les réponses des élèves de profil "Approximation" démontrent qu'ils ont évalué et comparé les pentes des sécantes en applicant un critère que souvent ils explicitent, plus ou moins correctement, sous la forme :

> pour des (incréments de) temps égaux, le mobile qui parcourt le plus grand espace a une plus grande vitesse.

Donc, nous avons observé que les élèves de profil "Approximation" avaient un schéma conceptuel de vitesse moyenne bien défini qui leur permettait de résoudre des situations locales ponctuelles avec des réponses tout à fait cohérentes.

c) Profil "Limite"

Les élèves auxquels nous avons assigné le profil "Limite" se caractérisent par un schéma conceptuel de la notion de vitesse instantanée où celle-ci est considerée comme la pente de la droite tangente au graphique au point consideré.

Nous en avons déduit qu'il s'est produit un changement conceptuel qui se manifeste par l'évidence de la construction d'un schéma conceptuel de la notion de vitesse instantanée, différentié du schéma conceptuel de la notion de vitesse moyenne, tant pour des mouvements considérés globalement que pour des intervalles partiaux de mouvements. En effet, on assiste à la naissance d'un nouveau vocabulaire qui incorpore les expressions ou les mots : "c'est la limite ...", "à ce moment (point) .. ", la "tangente", le "taux instantané", la "vitesse instantanée".

Voyons les expressions les plus courantes utilisées par les élèves pour définir la vitesse instantanée :

la pente de la droite tangente *indique* (*est /donne*) la vitesse en un point.

On peut observer qu'il s'agît là d'une définition indépendante des autres définitions de vitesse et qui se base sur la notion déjà connue de pente d'une droite et de celle récemment acquise de droite tangente à un graphique en un point.

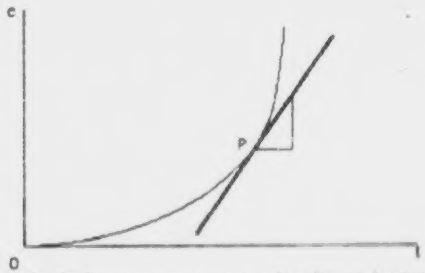
Quant au calcul de la vitesse instantanée sur le graphique du mouvement, nous constatons que la réaction immédiate de ces élèves consiste à dessiner la droite tangente et à calculer sa pente en prenant des mesures. Si on leur donne l'equation d'un mouvement rectiligne (par exemple $e(t) = 3t^2 + 4t$) et si on leur demande la valeur absolue de la vitesse à un moment donné (par exemple $t = 3$), ils réalisent un calcul dans le genre de celui de l'élève 2.62

$$c)\ \frac{d(t)-d(3)}{t-3} = \frac{(3t^2+4t)-39}{t-3} = 3t+13 \xrightarrow[t\to 3]{} \boxed{22\ \text{metros}/\text{seg}}$$

$$\begin{array}{rr|l} 3t^2 + 4t - 39 & & t-3 \\ \cline{3-3} -3t^2 + 9t & & 3t+13 \\ \cline{1-1} 13t - 39 & & \\ -13t + 39 & & \\ \cline{1-1} 0) & & \end{array}$$

où il faut tenir compte du fait que ces élèves n'ont pas étudié préalablement les limites. On voit donc que ces élèves n'ont pas une image symbolique unique de la notion de vitesse instantanée.

En ce qui concerne la représentation graphique, nous l'avons déjà dit, elle est claire: on dessine la tangente et on regarde sa pente :



Cette représentation servira également pour comparer des vitesses instantanées ainsi que pour déterminer les points de vitesse maximum.

Par conséquent, le schéma conceptuel des élèves de profil "Limite" consiste en une image graphique (la droite tangente et sa pente), une définition (la pente de la droite tangente), des propiétés (à plus grande pente, plus grande vitesse ; si la tangente est horizontale ou si la pente est nulle, la vitesse est nulle) et des procédés de calcul.

3- Evolution des profils au long de l'expérience d'apprentissage

La suite des séquences des entrevues nous a permis d'établir l'évolution des profils de ces élèves. Nous avons introduit un profil mixte "Primitif-Aproximation" pour décrire les élèves qui ont répondu aux deux premières entrevues d'après des modèles de ces deux profils, selon les questions. C'est ainsi que nous nous sommes plutôt trouvés avec des réponses correspondantes au profil "Primitif" dans les comparaisons de vitesses et des réponses correspondantes au profil "Approximation" dans les calculs et dans les représentations graphiques. Ces réponses incohérentes entre elles sont le symtôme d'un processus de perte d'équilibre et de restructuration du schéma conceptuel, ce qui explique le grand nombre de profils mixtes "Primitif-Aproximation" à la deuxième entrevue.

Nous présentons le cadre des profils et de leur évolution, dans lequel nous pouvons apprécier le grand nombre d'élèves (onze) qui ont atteint le profil "Limite" et comment ce processus n'est pas du tout uniforme. Il est intéressant aussi d'observer les trois élèves de profil "Approximation" à la fin de la phase d'apprentissage, qui sont donc restés à mi-chemin, mais qui ont démontré avoir des ressources pour résoudre des situations de vitesses instantanées. Plus inquiétants sont les quatre élèves qui maintiennent leur profil initial "Primitif", qui n'ont donc rien compris de ce qu'ils ont fait en classe, rien acquis au point de vue conceptuel.

↑ Profils

Limite			1 15 34 35 39 44 62 64 81 87 105
Aprox		1 15 35 44	21 83 93
Mixte Prim/ Aprox	39 87	21 34 39 64 81 93	
Prim.	1 15 21 30 34 35 38 44 53 62 64 79 81 83 93 105	30 38 53 62 79 83 105	30 38 53 79
	Entrevue 1	Entrevue 2	Entrevue 3

→ Entrevues

Cadre des élèves :

Lycée 1	1	15	21	30	34	35
Lycée 2	38	39	44	53	62	64
Lycée 3	79	81	83	87	93	105

4 - Conclusions

Pour finir, nous voulons remarquer que les **profils** des élèves se sont révélés comme un instrument utile qui nous ont permis :

- de determiner les préconceptions des élèves au moment d'aborder un nouveau sujet
- de comprendre les difficultés et les erreurs des élèves
- de connaître et d'intervenir dans le processus dynamique de l'évolution des élèves au cours d'une étape d'apprentissage.

Notes :

1. Cet exposé est le résumé d'un chapitre de la thèse de doctorat "La velocidad : introducción al concepto de derivada" présentée par l'auteur à la Faculté des Sciences de l'Universitat Autònoma de Barcelona (Décembre 1990) et dirigée par le professeur D. Johnson.

2. Je donne à "schéma conceptuel" le sens de "concept image" défini par Tall et Vinner (1981).

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Students' Mental Prototypes for Functions and Graphs

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This research study investigates the concept of function developed by English A-level students. The hypothesis is that students develop prototypes for the function concept in much the same way as they develop prototypes for concepts in everyday life. The definition of the function concept, though given in the curriculum, is not stressed and proves to be inoperative, with student understanding of the concept reliant on properties of familiar prototype examples. Investigations reveal significant misconceptions. For example, threequarters of a sample of students starting a university mathematics course considered a constant function is not a function in at least one of its graphical or algebraic forms, and threequarters thought that a circle is.

The concept of a function begins in the U.K. National Curriculum at around the age of nine (algebra attainment target 6, level 3, DES 1989), yet the *definition* of a function causes great difficulties later in the curriculum. The idea of *defining* a concept is at variance with a child's everyday experience where a perceived concept such as "bird" is developed through encounters which begin to focus on salient features. "That is a bird. ... It flies, ... has *wings* ... and *feathers* ... and a *beak* ... and *lays eggs*". New creatures against these various critieria: "Is a chicken a bird?" ..."It has wings, feathers, a beak and lays eggs, but it doesn't fly. OK, some birds don't fly. We will say a chicken is a bird." "Is a bat a bird?" "It flies and has wings, but it is really a flying mouse, so it is not a bird." Over a period of time the individual builds a complex of interconnected prototypes which help to test whether newly encountered examples are instances of the general concept. (Smith 1988). It therefore comes as no surprise that students are likely to apply similar criteria when faced with concepts in the mathematics class.

We hypothesize that the students develop "prototype examples" of the function concept, such as: a function is like $y=x^2$, or a polynomial, or $1/x$, or a sine function. When asked if a graph is a function, in the absence of an operative definition, the mind attempts to respond by resonating with these mental prototypes. If there is a resonance, the individual experiences the sensation and responds positively. If there is no resonance, the individual experiences confusion, searching the mind to formulate the reason for failure to obtain a mental match. Positive resonances may be in error if they evoke inappropriate properties of prototypes, for instance, that a function is (usually) described by a formula, or that the graph (such as a circle) looks familiar. Negative resonances may also be in error, for instance strange looking graphs may not be considered functions, or a constant may not be thought of as a function because previously encountered prototypes depend on a variable, and so must vary.

Students' conceptions of a function

Following ideas of gathering evidence about student conceptions of functions in Vinner (1983) and Barnes (1988), we asked a group of twenty eight students (aged 16/17) to:

> Explain in a sentence or so what you think a function is.
> If you can give a definition of a function then do so.

They had studied the notion of a function during the previous year and had used functions in the calculus but with little emphasis on the technical aspects of domain, range and so on. None gave satisfactory definitions, but all gave explanations, including the following:

- a function is like an equation which has variable inputs, processes the inputted number and gives an output.
- a process that numbers go through, treating them all the same to get an answer.
- an order which plots a curve or straight line on a graph.
- a term which will produce a sequence of numbers, when a random set of numbers is fed into the term.
- a series of calculations to determine a final answer, to which you have submitted a digit.
- a set of instructions that you can put numbers through.

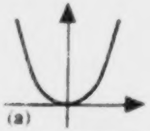
The majority expressed some idea of the *process* aspect of function – taking some kind of input and carrying out some procedure to produce an output – but no one mentioned that this only applies to a certain *domain* of inputs, or that it takes a *range* of values. Many used technical mathematical words, such as *term, sequence, series, set,* in an everyday sense, intimating potential difficulties for both students and teachers in the process of transferring mathematical knowledge.

Graphs of functions

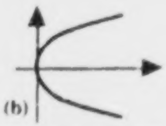
School mathematics is intended to give students experiences of mathematical activities, rather than plumb the formal depths of logical meaning. The formalities may be mentioned, but they are not stressed because they do not appear to be appropriate until the student has a suitable richness of experience. But the collection of activities inadvertently colours the meaning of the function concept with impressions that are different from the mathematical meaning which, in turn, can store up problems for later stages of development.

To investigate students' concept images of the function concept, we showed the same 28 sixth-formers, and one hundred and nine students starting their first year of university mathematics, nine graph sketches and asked:

Which of the following sketches could represent functions? Tick one box in each case. Wherever you have said no, write a little explanation why by the diagram.



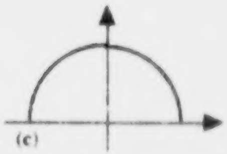
	% yes	% no
school	100	0
univ	97	3



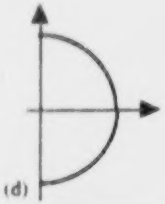
	% yes	% no
school	95	4
univ.	80	20

These first "starter" questionsgive little information and we can set little store by them. With the "usual conventions", that the horizontal axis represents the independent variable and the vertical axis the dependent variable, (a) would be adjudged correct and (b) false, (as indicated by printing in heavy type); equally (b) could be true if it represented x as a function of y. Only two university students interpreted (b) in the latter manner, one saying "look at it a different way", the other "$f(y)=x$". The university students responding negatively to (b) often did so with a comment equivalent to the fact that this "sometimes has two y's for each x".

However, these responses are seen in new light when compared to responses to similar questions using semicircles instead of parabolas:



	% yes	% no
school	61	36
univ.	91	9



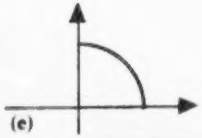
	% yes	% no
school	43	57
univ.	70	28

There is a drop to 61% of school pupils thinking figure (c) is a function whilst it seems that more (57%) now correctly respond that figure (d) is not. The drop in belief in figure (c) compared with (a) was accompanied with comments such as:

> "if a function the graph would continue, not just stop", "stops dead, values are not limitless", "the lines would have to continue", "this could not apply to any value".

The word "continuous" is here used with the everyday meaning of "continuing without a break". Several of the explanations allude to ideas which suggest that functions should not be unnaturally curtailed. One student extended the graph to "continue" it for more values of x. As all functions studied by the students (polynomials, trigonometric functions etc) are defined by a formula, this suggests a prototype of a function being "naturally defined everywhere the formula is defined", leading to unease with "artificial" functions such as the top half of a circle.

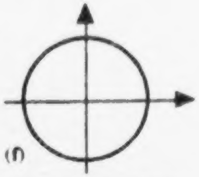
When faced with a quadrant of a circle (e), even fewer school pupils considered it a function. Their belief in a graph being a function through pictures (a), (c), (e) drops from 100% to 61% to 29% as the graph passes from parabola to semicircle to quadrant, becoming less familiar and restricted to a smaller and smaller domain. Later discussion revealed students who insisted the graph (e) was "not complete" because it was only *part* of a circle. For such a student *a function is a natural totality given by a formula, and it is essential to have it all, not an unnaturally selected part.*



	% yes	% no
school	29	61
univ.	*not asked*	

Although a quadrant of a circle (which is the graph of a function) is considered not to be a function by most pupils, the situation is reversed with a complete circle. Approximately two thirds of the students in school and university considered the circle in figure (f) to be a function.

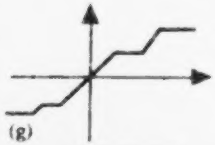
Those thinking it was not a function included two from school saying, "You can't work a function that goes back on itself" or the "equation is $x^2+y^2=25$", implicitly – but not explicitly – suggesting that y is not determined uniquely by x. Most of the negative responses from the university students appropriately alluded to the idea that each value of x might be related to more than one y.



	% yes	% no
school	64	29
univ.	65	35

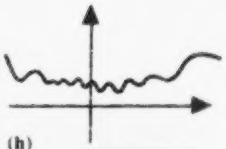
But why do two thirds of the students think that the circle is a function? The term "implicit function" (or "many-valued function") is often used (incorrectly) in Britain, so the students may have heard the term "function" used in this context. There is also the familiarity of the graph and its formula which may resonate with some of the function prototype properties.

The final three pictures – (g), (h) and (i) – presented even more conflict to students. Both (g) and (h) could satisfy the function definition, but not (i) because there is a part of the graph where one value of x corresponds to more than one value of y. However, they look strange, so none of them fit the students' mental collection of prototypes. In general the university students appear to cope better with these more general curves. The school pupils greater success with (i) is an illusion, due to its unfamiliarity rather than any formal property of a function:



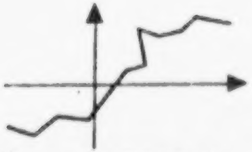
(g)

	% yes	% no
school	**50**	32
univ.	**91**	8



(h)

	% yes	% no
school	**14**	79
univ.	**72**	26



(i)

	% yes	% no
school	11	**82**
univ.	39	**58**

> "graphs are usually smooth, either a straight line or curve, not a combination of the two, nor staggered, when dealing with a function", "these are absurd", "too complicated to be defined as a function", "too irregular", "no regular pattern".

None of these three graphs match the students' mental collection of function prototypes. Their comments support the hypothesis that their prototypes are usually "given by a formula", which tend to have a "recognizable shape", a "smooth" graph, seem "regular", and so on.

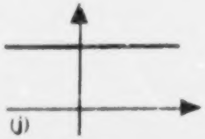
Three school pupils do focus on the part of the graph where there are three y-values for each x-value: "here the curve goes back on itself", ... "there is an irregular peak which could not be created from a function". They are beginning to evoke the restriction that each x should have only one y. But they do not apply this test consistently in the earlier examples, and their "definitions" of a function are:

> "a mathematical command or identity",
> "an equation with a variable factor ... e.g. $f(x)=x+2$",
> "the product of a series of numbers which the numbers must undergo".

Not one of the school pupils consistently evoked a coherent function concept. Only eight of the university students (7% of the total) gave a consistent set of replies to all the graphs, with one other consistently allowing x to be a function of y as well as y being a function of x.

Constant functions seemed to cause significant problems, so graph (j) was given to the university students, but not to those at school (in lieu of graph (e) above):

Almost half the students at university considered a constant is not a function, often because y is independent of the value of x. Such an interpretation is sometimes used in certain contexts, for instance, asserting that $dy/dx=1/x$ describes dy/dx as a function of x but not of y. Clearly implicit in school mathematics is that the notion of a function has variables, and if a variable is missing, then the expression is not a function of that variable.



(j)

	% yes	% no
school	*not given*	
univ.	55	44

Algebraic expressions as functions

To consider the meaning of a function in terms of formulae (as in Barnes, 1988), we asked the university students to say which of a number of symbolic expressions or procedures could represent y as a function of x. Some of these were algebraic equivalents of the pictorial representations mentioned earlier. Thirty eight of the 109 students explicitly mentioned at least once word "many-one" or that for each x there must be one y, or equivalent. We include separate columns representing the responses of these 38 "more knowledgeable" students. Once again the expression $y=x^2$ is almost universally regarded as a function, but $y=4$ is not. As in Barnes (1988), a majority of all students consider the circle $x^2+y^2=1$ to be a function. Although those showing more technical knowledge perform better, still only 47% think that $y=4$ is a function whilst only 60% think $x^2+y^2=1$ is not.

	University students (N=109)		Subset showing more knowledge (N=38)	
	% yes	% no	% yes*	% no*
(1) $y=x^2$	**96**	4	**95**	3
(2) $y=4$	**30**	69	**47**	53
(3) $x^2+y^2=1$	62	**37**	40	**60**
(4) $y=3/x$	**91**	9	**84**	16
(5) $xy=5$	**82**	17	**82**	18
(6) $y=\pm\sqrt{4x-1}$	67	**33**	34	**66**
(7) $y=\begin{cases} 0 & \text{if } x\leq 0 \\ x & \text{if } 0\leq x\leq 1 \\ 2-x & \text{if } x>1 \end{cases}$	**92**	7	**95**	5
(8) $y=0$ if x is a rational number	50	48	**42**	58
(9) $y=0$ (if x is a rational number), $y=1$ (if x is an irrational number).	75	22	**79**	21

Table 1

Expressions (4) and (5) show that the majority of students see $y=3/x$, $xy=5$ as functions, the major obstacle for the first being that it is not defined for $x=0$, and for the second, not only is it not defined for $x=0$, but the expression is not considered a function until it has been manipulated to get "y as an expression involving x". The latter is a common function prototype.

Expression (6) shows that the majority of students think that $y=\pm\sqrt{4x-1}$ is a function. It resonates with the "y equals an expression in x" prototype even though y is not given uniquely. The "more knowledgeable" minority show a marked improvement because they are consciously aware that a function must give (at most) one value of y for each value of x.

Expressions (7), (8) and (9) address the problems of defining functions differently on different sub-domains. These do not fit the prototypes familiar to most students. The correct response to (7) is remarkably high given that Vinner (1983) and others have found that function prototypes usually involve only a single formula. *No student made such a comment*, perhaps because in this case each formula on the subdomains is familiar and the function is everywhere defined. The fact that (8) is not everywhere defined definitely caused problems because:

> "y is not defined for all x", or "doesn't state what y is if x is not rational".

Expressions (8) and (9) provoke difficulties because they fail to fit the students' prototypes.

> "no real link with x, i.e. not actually applying a function to x, where the answer would be y", "y is not in proportion to x", "no relation between x and y", "not continuous on the real number line".

Conflicts with constant functions

Comparing the student responses to the *expression* $y=4$ and the *graph* of y=constant, only 28% reply (correctly) in the affirmative to both. Table 2 correlates the responses (with the "more knowledgeable" subset percentages in brackets).

Is y=const a function?		algebra	
		% yes	% no
graph	% yes	**28 (42*)**	29 (26*)
	% no	3 (5*)	41 (26*)

Table 2

There is evidence of conflict in a significant number of scripts, as students change their mind when realizing that the algebraic expression clearly does not involve x, but the graph seems more likely to be a function. One student who thought initially that $y=4$ was not a function, then wrote it as $y=4x^0$, hence obtaining "a formula involving x". This may very well be related to the description of the relationship between x and y in terms of variables: that the dependent variable y varies as the independent variable x varies. The expression $y=4$ offends this prototype because y does not vary!

The circle as a function

Comparing the responses to the graphic and algebraic representations of a circle, we find that 52% erroneously regard both graph and expression as representing functions, 12% say "yes" to graph and "no" to expression, 10% say "no" to graph and "yes" to expression, and only 25% correctly say "no" to both (table 3). The more technical responses increase the percentage correct from 25% to 47% – still less than half.

Is a circle a function?		algebra	
		% yes	% no
graph	% yes	52 (18*)	12 (24*)
	% no	10 (11*)	**25 (47*)**

Table 3

The position is worse when we consider which students give a correct response to both questions in algebraic and graphic modes:

> Only 11% of all students assert both that y=constant is a function and a circle is not. The percentage only increases to 29% among the more technical responses.

Thus, even amongst the most able students in the sixth form, the vast majority do not have a coherent concept of function at the end of their A-level studies.

Reflections

Because the general function concept is difficult to discuss in full generality, many teachers take the pragmatic route of de-emphasizing theory and emphasizing practical experience. Attempts to teach the formal theory, as in the New Mathematics of the sixties, have proved unsuccessful. But the other side of the coin – teaching the concept through examples, as in the current curriculum – leads to mental prototypes which give erroneous impressions of the general idea of a function. Even amongst the students who receive some training in the notion of a function, only a small minority respond coherently and consistently.

We seem to face a formidable, fundamental obstacle:

> The learner cannot construct the abstract concept of function without experiencing examples of the function concept in action, and they cannot study examples of the function concept in action without developing prototype examples having built-in limitations that do not apply to the abstract concept.

It is the awareness of this obstacle which should be a major focus of future research to help students cope with the fundamental and necessary mental reorganisation which accompanies the construction of mental imagery for mathematical definitions.

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ILLUSTRATIONS DE PROBLÈMES MATHÉMATIQUES COMPLEXES METTANT EN JEU UN CHANGEMENT OU UNE SÉQUENCE DE CHANGEMENTS PAR DES ENFANTS DU PRIMAIRE

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Our previous research showed how difficult it is for children to solve problems involving reconstruction of a change, and to perceive change in the external representations generally proposed in the teaching of mathematics to illustrate these situations. Graphic codes and conventions which are in use, with the intention of recreating this process, are mostly misread in a static way by the children. The present study intends chiefly to improve our knowledge of the representations developed by children (6 to 12 years) in a context of complex problems involving change or sequence of changes. 180 children (30 children at each grade level) were invited to represent and solve different problems involving reconstruction of a change. Results reveal the abundance of codes used by children to illustrate these problems, and put in light the evolution from grade 1 to 6 in the representation and conception of these complex situations.

Plusieurs situations en mathématiques font appel à la représentation d'un changement. Il en est ainsi par exemple des situations où une certaine entité (collection, grandeur, position...) est soumise à un ou plusieurs changements successifs (transformations ou déplacements), passant d'un état initial donné à un certain état final. La reconstruction mentale du processus requis pour résoudre ces situations, lorsque par exemple un état initial et final sont donnés et qu'il faut déduire le changement qui s'est opéré, s'avère complexe pour les jeunes enfants (Vergnaud, 1976; Carpenter, Moser, 1982; Riley, Greeno, 1983; Resnick, 1982; Bednarz, Janvier, 1983, 1987...). Quelques études montrent que ces difficultés ne sont pas spécifiques aux jeunes enfants, et se retrouvent en fait à tous les niveaux du primaire (Vergnaud, 1976; Conne, 1979; Bednarz, Schmidt, Janvier, 1989...). Ainsi, lorsque les situations se complexifient et mettent en jeu des séquences de changements successifs, les mêmes procédures erronées réapparaissent, exprimant dans plusieurs cas une centration sur les états (Bednarz, Schmidt, Janvier, 1989).

L'apprentissage de ces situations complexes en mathématiques prend appui, à différents niveaux scolaires, sur un certain nombre de signifiants (illustrations, schémas, diagrammes...) qui cherchent à recréer, via un ensemble de codes et conventions graphiques, un processus de changement (flèches de transformations, de déplacements sur la droite numérique, indices d'action illustrant un ajout ou un retrait d'éléments dans une collection donnée...). Or nos recherches passées montrent que les élèves ne perçoivent nullement le changement dans ces représentations qui leur sont couramment proposées dans l'enseignement, et que celles-ci s'avèrent peu un support à sa reconstruction (Bednarz, Janvier, 1985).

Problème posé par la représentation de situations mathématiques mettant en jeu un changement

Plusieurs études (Mary, 1983; Campbell, 1981; Bednarz, Janvier, 1985, 1986; Girardon, Janvier, 1987...) apportent un éclairage au problème soulevé par la représentation de telles situations. Ainsi, certains travaux ont cherché à rendre compte de l'interprétation que les étudiants donnent à des «représentations dynamiques» voulant illustrer un processus de changement. Claudine Mary (1983), dans son travail sur le «film et l'enseignement des mathématiques: analyse théorique et expérimentation» a montré que la majorité des élèves du secondaire sont incapables de décrire des déplacements ayant été présentés dans un film animé à l'écran. Ils sont quelquefois attirés par un mouvement incident ou sont incapables de situer le mouvement d'un élément par rapport au système. L'étude de Patricia Campbell (1981), portant sur l'interprétation des images utilisées en enseignement des mathématiques pour illustrer une transformation d'une collection donnée (action d'enlever ou d'ajouter), où l'on cherche à recréer l'action en ayant recours à certains codes graphiques conventionnels, nous révèle que la tranformation est loin d'être perçue par de jeunes enfants. Girardon et Janvier (1987) confirment également que certains types de graphiques en sciences, cherchant à illustrer une séquence de changements, sont difficiles à interpréter par les élèves.

Les travaux de Janvier, Bednarz, Bélanger (1986) mettent bien en évidence à cet effet l'écart qui existe entre les intentions d'une représentation (intentions du concepteur de manuel ou de l'enseignant) et les interprétations fournies par l'élève dans le traitement de ces représentations. C'est ainsi, par exemple, que la flèche de déplacement (sur la droite numérique) sera avant tout interprétée comme une flèche pointeur (désignant un nombre) ou majoritairement comme une corde ensembliste, délimitant plusieurs points (Bednarz, Janvier, 1985).

D'autres études, davantage centrées cette fois sur l'analyse des illustrations, schémas... utilisés, permettent de rendre compte des codes et consignes qui ont pour but dans la représentation externe de recréer le changement. Ainsi l'étude de Friedman et Stevenson (1980) permet de rendre compte de quatre principaux indicateurs de changement dans les codes utilisés à travers différentes cultures et différentes époques de l'histoire: la description d'une action en cours (par le recours à des indices de posture des personnages, de déséquilibre des éléments), la description d'un même objet à différents moments successifs indiquant un changement dans le temps, l'utilisation de métaphores picturales, le recours à des codes symboliques (tels les flèches...). Ces indicateurs de changement apparaissent très différents de ceux que l'on retrouve dans les représentations externes habituellement utilisées dans l'enseignement des mathématiques. Les recherches présentées précédemment éclairent d'une part la façon dont sont vus par les élèves les codes et conventions qui ont pour but, dans l'enseignement, de recréer le changement, et permettent d'autre part de distinguer, dans le dynamique figé des représentations, les différents codes possibles qui sont utilisés pour recréer celui-ci.

Notre connaissance des difficultés des élèves à, d'une part, résoudre certaines situations mathématiques complexes mettant en jeu un changement ou une séquence de changements, et d'autre part à interpréter les représentations externes fournies a priori dans l'enseignement qui cherchent à recréer un tel processus, nous ont conduit à nous attarder davantage à l'étude des représentations externes développées par les enfants dans ces situations.

Objectifs de la recherche

Le but de cette recherche est avant tout de mieux connaître les représentations développées par les enfants dans un contexte de résolution de problèmes mathématiques mettant en jeu les signifiés dynamiques de transformation ou déplacement à reconstruire.

- Comment se caractérisent les illustrations, schémas... développés à chaque niveau scolaire (1ère à 6e année, 6 à 12 ans) pour illustrer un changement et une séquence? Comment ces représentations évoluent-elles d'un niveau scolaire à l'autre? Quelles *conceptions des relations sous-jacentes* aux situations-problèmes présentées traduisent-elles?
- Quels sont les codes et conventions graphiques que les enfants utilisent spontanément pour illustrer le dynamisme du changement ou d'une séquence? Comment ces codes se comparent-ils à ceux habituellement utilisés dans l'enseignement?
- Y a-t-il un lien entre le type de représentation développée et la performance à résoudre ces problèmes?

Voilà quelques unes des questions qui sous-tendent la recherche que nous avons conduite auprès d'enfants du primaire (6 à 12 ans).

Méthodologie

À partir d'un cadre de référence reposant sur une analyse des problèmes de structure additive (Vergnaud, 1976), une banque de situations problèmes a été élaborée tenant compte des variables suivantes:

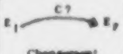
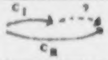
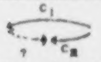
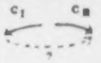
Structure du problème: problèmes mettant en jeu un changement simple ou une séquence de changements plus ou moins complexe, directe ou non (le changement résultant pouvant être dans le sens du changement initial ou non).

Nature du changement: transformation ou déplacement.

Nature des états sur lesquels agit le changement: collections discrètes ou grandeurs continues.

Le tableau qui suit (tableau 1) présente la répartition des problèmes expérimentés à chaque niveau, qui mettaient en jeu la reconstruction d'un changement.

Tableau 1
Répartition des problèmes présentés selon la structure sémantique sous-jacente

Niveau / Structure sémantique	1ère	2e	3e	4e	5e	6e
 Changement simple	x	x				
 Séquence directe de changements	x	x	x	x	x	
 Séquence indirecte de changements			x	x	x	
				x	x	x

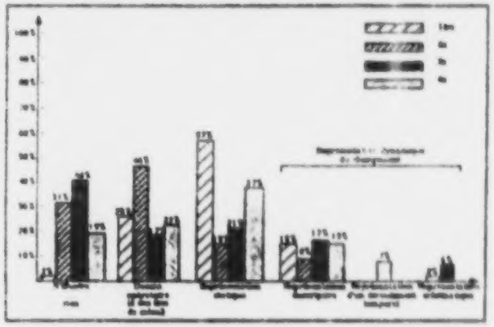
Ces situations ont été expérimentées dans une école de milieu socio-économique moyen de l'île de Montréal (≈ 30 enfants par niveau de la 1ère à la 6e année). Les enfants avaient pour tâche dans un premier temps de représenter le problème, puis dans un deuxième temps, de résoudre le problème.

Résultats

Tous les dessins produits par les enfants ont donné lieu à un codage systématique, et des analyses classificatoires ont été réalisées permettant d'établir une typologie des représentations produites par les enfants. Ainsi, dans le cas de situations mettant en jeu un changement simple, l'analyse permet de discerner les représentations à caractère dynamique (qui cherchent à rendre compte d'un changement) *des représentations statiques*, simple illustration des objets décrits dans la situation (*statique-objet*) ou illustration d'un état figé de la situation (*statique-état*). Parmi les *représentations dynamiques*, l'analyse permet de différencier plusieurs façons d'illustrer le processus de changement: certains enfants produisent *une représentation descriptive de la situation*, illustrant sur une même image l'état initial et l'état final, une action en cours, ou la répétition de l'objet transformé, avec dans tous les cas recours à des codes descriptifs issus de la situation. D'autres enfants cherchent à illustrer un déroulement temporel par une suite d'images décrivant la situation qui évolue (*succession d'états*). D'autres enfin ont recours à une *représentation* davantage *schématique*, en utilisant des codes graphiques empruntés aux bandes dessinées ou des codes symboliques (tels les flèches...).

Par ailleurs, l'analyse met en évidence une évolution importante d'un niveau scolaire à l'autre dans la représentation de ces situations. D'une représentation majoritairement statique en 1ère et 2e années, celle-ci évolue vers une représentation dynamique, au départ descriptive de la situation. Celle-ci fera graduellement place chez les enfants plus âgés à une représentation plus schématique, avec recours à des codes symboliques (cf. figure 1).

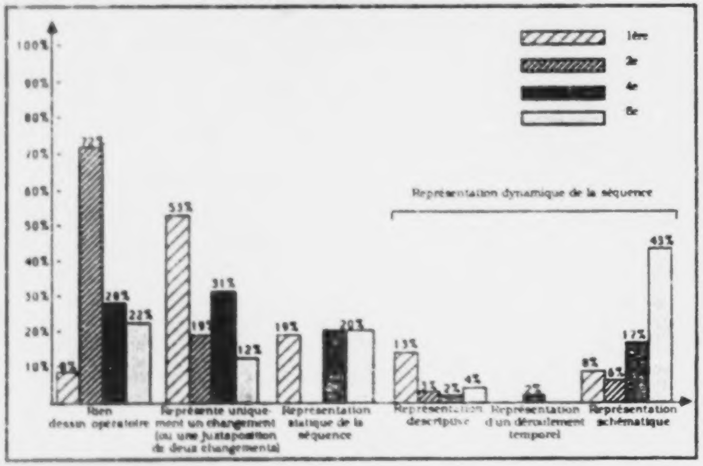
Figure 1
Évolution dans le type de représentation du changement en fonction du niveau scolaire



Nos résultats montrent par ailleurs que certaines situations prêtent plus facilement que d'autres à une représentation externe du changement; il en est ainsi des situations faisant appel à un changement de type «temporel» (dans ce cas, le processus de changement est davantage caractérisé par un déroulement temporel de nature continue). Les situations mettant en jeu un changement de type procédural (cette distinction entre les aspects temporels et procéduraux des transformations est due aux chercheurs Girardon-Morand et Janvier, 1987) apparaissent moins aisées à représenter (le processus est ici caractérisé par une intervention plus ponctuelle produite sur un état donné et relève en quelque sorte d'une procédure entre deux états).

Quant à la représentation externe d'une séquence de changements, celle-ci apparait beaucoup plus complexe pour l'enfant et ne se retrouve présente que chez les enfants les plus âgés (cf. figure 2).

Figure 2
Évolution dans le type de représentation de la séquence en fonction du niveau scolaire



Enfin, les résultats révèlent des liens intéressants entre les schémas développés par les enfants pour illustrer les situations mettant en jeu la reconstruction d'un changement et leur performance à résoudre de tels problèmes. Ainsi, les enfants qui échouent à ces problèmes produisent majoritairement une représentation statique (ils illustrent un état de la situation), alors que les «reconstructeurs» illustrent le changement.

Conclusion

À travers cette étude, des différences importantes apparaissent d'un niveau à l'autre dans la façon dont l'enfant se représente ces situations. Des liens ont été mis en évidence entre la représentation externe que l'enfant produit du changement et sa performance à résoudre des problèmes complexes mettant en jeu la reconstruction d'un changement. La représentation externe rend ainsi compte des schèmes qui modulent les actions des enfants dans ces situations. En particulier, cette étude révèle chez les «non-reconstructeurs» une conception majoritairement statique des relations sous-jacentes.

Enfin, les résultats de cette recherche, menée auprès d'enfants de 1ère à 6e année du primaire (6 à 12 ans) mettent en évidence une richesse de codes utilisés par les enfants pour illustrer le changement, qui vont bien au-delà des codes habituellement utilisés dans l'enseignement des mathématiques pour illustrer une transformation ou un déplacement. Ces codes et conventions graphiques nous fournissent des indices de représentations transitoires sur lesquelles un apprentissage pourrait s'appuyer.

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THE OPERATOR CONSTRUCT OF RATIONAL NUMBER: A REFINEMENT OF THE CONCEPT

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Some results of an ongoing mathematical/semantic analysis of the additive and multiplicative conceptual fields being conducted by the Rational Number Project are presented. These results suggest refinements of the operator construct of rational number, one of which presented herein, is called duplicator/partition-reducer (D/PR). The analysis uses two notations: a generic manipulative aid and a generalized notation for mathematics of quantity. As D/PR, a rational number such as 3/4 is a 3-for-4 exchange function. Implications of this for computational procedures and problem solving are suggested and presented.

A mathematical/semantic analysis of the subconstructs of rational number (Kieren 1976) conducted by the Rational Number Project (Behr, Harel, Post, & Lesh, 1990, in press) has led to a refinement and a deeper understanding of these constructs. Behr et al. (in press) imposed different combinations of interpretations on the numerator and denominator and hypothesized 5 distinct interpretations of the operator construct: Duplicator/Partition-Reducer, Stretcher/Shrinker, Multiplier/Divisor, Stretcher/Divisor, and Multiplier/Shrinker. Extensive analysis of the first two has identified interesting similarities and differences. The purpose of this paper is to report some findings about the Duplicator/Partition-Reducer (D/PR) interpretation.

The numerator as a duplicator suggests that its effect on an operand quantity is to take it as a single entity and make a number of copies of that quantity so the total number equals the numerator. The denominator as a partition-reducer suggests that its effect on an operand is to partition that whole quantity into a number of parts equal to the denominator and then to adjust the quantity to the size of one of its parts. Both the duplicator and partition-reducer operate on the operand quantity as a whole (a unit), not on the points or objects that make up that whole. On the other hand, stretcher and a shrinker interpretations do operate on the points or objects that comprise the whole (Behr et al., in press).

We have developed two notational systems to conduct, and communicate results of, these analyses: a generic manipulative aid notation and a generalized mathematics of quantity notation. Both indicate the type of units we hypothesize children would need to form in order to understand the particular concept under analysis.

Representations using the generic manipulative aid suggest object manipulations through which a child should be guided to provide an experiential base for understanding. We present the notation needed to interpret some of the analysis of the D/PR interpretation:

1. From a single object, 0, we can conceptualize a *singleton unit*, denoted in the manipulative aid and mathematics of quantity notational systems respectively as: (0), 1(1-unit); or, we can conceptualize several singleton units: (0) (0) (0), 3(1-unit)s.
2. From several objects, 0 0 0, we can conceptualize a *composite unit*: (0 0 0), 1(3-unit).
3. From several singleton units, (0) (0) (0) (0), or several composite (3-unit)s, (0 0 0) (0 0 0) (0 0 0) (0 0 0), we can conceptualize a *unit-of-units*:
   ((0) (0) (0) (0)), 1(4(1-unit)s-unit), or
   ((0 0 0) (0 0 0) (0 0 0) (0 0 0)), 1(4(3-unit)s-unit).
4. From several composite units-of-units,
   ((00)(00)(00)(00)) ((00)(00)(00)(00)) ((00)(00)(00)(00)),
   we can conceptualize a composite *unit-of-units-of-units*:
   (((00)(00)(00)(00)) ((00)(00)(00)(00)) ((00)(00)(00))(00))),
   1(3(4(2-unit)s-unit)s-unit).

We will use 3/4 applied to 8 in all illustrations. In Figure 1 a manipulative aid representation of 3/4 as a composite of the operators 3/1 and 1/4 is given. It suggests that 3/1 and 1/4 are 3-for-1 and 1-for-4 exchange functions; and consequently, that 3/4 is a 3-for-4 exchange function.

Figure 1. Three-fourths as a composite of the operators 3/1 and 1/4.

a. (0) (0) (0) (0) (0) (0) (0) (0)	a. 8(1-unit)s.
b. $\begin{pmatrix}0&0&0&0\\0&0&0&0\end{pmatrix}$	b. Reunitize to 1(8-unit) in order to form an entity which can be duplicated.
c. $\begin{pmatrix}0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\end{pmatrix}$	c. Apply the 3-for-1 duplicator function to duplicate 1(8-unit) to 3(8-unit)s.
d. $\left(\begin{pmatrix}0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\end{pmatrix}\right)$	d. Unitize the 3(8-unit)s into a unit-of-units: 1(3(8-unit)s-unit).
e. $\left(\left(\begin{array}{ccc\|c}0&0&0&0\\0&0&0&0\end{array}\right)\left(\begin{array}{cc\|cc}0&0&0&0\\0&0&0&0\end{array}\right)\left(\begin{array}{c\|ccc}0&0&0&0\\0&0&0&0\end{array}\right)\right)$	e. Partition the 1(3(8-unit)s-unit) into 4 parts.
f. $\left(\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}\right)$	f. Reunitize the partitioned unit-of-units into 1(4(6-unit)s-unit) to prepare for the 1-for-4 reducer.
g. 0 0 0 0 0 0	g. Apply a 1-for-4 reducer to change 1(4(6-unit)s-unit) to 1(6-unit).
h. (0) (0) (0) (0) (0) (0)	h. Reunitize 1(6-unit) to 6(1-unit)s.

A manipulative representation of 3/4 as a composite of 3/1 and 1/4 varies in complexity depending on the order in which 3/1 and 1/4 are applied. The partition to go from step d to e (Figure 1) is more complex with 3/1 applied first because of perceptual distractors, reunitizing the 3(8-unit)s to (24-unit)s and then to 4(6-unit)s might be less complex. A demonstration of this, one with 1/4 applied first, and a mathematics of quantity representation which matches Figure 1 in a step-for-step manner appears in (Behr et al., in press).

In Figure 2 a manipulative aid representation of 3/4 of 8 with 3/4 as a direct 3-for-4 exchange function rather than as a 3/1-and-1/4 composite function is given.

Figure 2. Three-fourths as a direct 3-for-4 exchange.

a. (0) (0) (0) (0) (0) (0) (0) (0)	a. 8(1-unit)s.
b. (00) (00) (00) (00)	b. Reunitize 8(1-unit)s into 4(2-unit)s to prepare for a 3-for-4 exchange.
c. (0 0) (0 0) (0 0)	c. 3(2-unit)s are exchanged for 4(2-unit)s.
d. (0 0 0 0 0 0)	d. 3(2-unit)s are reunitized to 1(6-unit).
e. (0) (0) (0) (0) (0) (0)	e. 1(6-unit) is unitized to 6(1-unit)s.

Application of the D/PR Interpretation to Computation Algorithms.

The D/PR exchange function interpretation of rational number is very powerful. It provides a close relationship between a manipulative level interpretation of a rational number and the syntax of the corresponding mathematics-of-quantity symbolic representation (Behr et al., in press). The manipulative representations suggest algorithmic computation procedures for the arithmetic-of-numbers. At the manipulative level a 3-for-4 exchange of objects in an application of the D/PR concept of 3/4 to some operand can be accomplished as follows: (a) arrange the objects of the operand into 4 groups and then (b) replace these 4 groups by 3 groups of the *same size*. This generalizes to all rational numbers in the obvious way. The procedure carries over to a symbolic representation in exactly the same way; that is, to apply 3/4 as a D/PR to 8, first rewrite 8 as 4•2, then replace the 4 in this expression with a 3, to give 3•2. For example:

3/4(8) = 3/4(4•2) (Partition 1 group of 8 into 4 groups.)
= 3•2 (Replace 4 groups of 2 with 3 groups of 2.)
= 6 (Reunitize 3 groups of 2 to 6 groups of 1.)

The manipulative display (Figure 2) which matches the symbolic form above suggests the 3/4 function rule, exchange 4 groups with 3 groups. What the adult reader might think of as cancellation of the 4 in 3/4 with the 4 in 4•2 is, as suggested in Figure 2, is an application of the concept of 3/4 as a 3 groups-for-4 groups exchange function. The 3/4 in the notation 3/4(4•2) is a function rule. The rule is to exchange the 4 in 4•2 with a 3.

Thus, application of the function 3/4 to the preimage number 8 (after being expressed as 4•2) gives the image value 6 (expressed as 3•2).

The notation 3/4(8) is ordinary function notation. More precisely, 3/4(8) means the value of the function 3/4 evaluated at 8. With this interpretation of the rational number, the expression 3/4 times 8 really does mean 3/4 of 8.

With 3/4 interpreted as a composition of a 3-for-1 and a 1-for-4 exchange, as in Figure 1, or as the a 1-for-4 and 3-for-1 (not demonstrated with the manipulative aid in this paper), arithmetic-of-number computation algorithms are suggested as shown in Figure 3. The symbolic representation in Figure 3a corresponds almost step-for-step to the manipulative representation in Figure 1; Figure 3b shows an additional step in going from 3•8 to 4•6 which reflects an alternate unitization to that shown in Figure 1; Figure 3c shows the computation algorithm for 3/4 interpreted as a 3-for-1 exchange applied to the result of a 1-for-4.

Figure 3. Computation procedures: 3/4 as a composite of 3/1 and 1/4.

a. $\frac{3}{4}(8) = \frac{1}{4}\left(\frac{3}{1}(1\cdot 8)\right)$	3/4 expressed as composite of 1/4 and 3/1 and 8 groups of 1 as 1 group of 8.
$= \frac{1}{4}(3\cdot 8)$	Replace 1 group of 8 with 3 groups of 8.
$= \frac{1}{4}(4\cdot 6)$	Reunitize the 3 groups of 8to 4 groups of 6.
$= 1\cdot 6$	Replace 4 groups of 6 with 1 group of 6.
$= 6$	Reunitize 1(6-unit) to 6(1-unit)s.
b. $\frac{3}{4}(8) = \frac{1}{4}\left(\frac{3}{1}(1\cdot 8)\right)$	3/4 expressed as composite of 1/4 and 3/1 and 8 groups of 1 as 1 group of 8.
$= \frac{1}{4}(3\cdot 8)$	Replace 1 group of 8 with 3 groups of 8.

Figure 3 is continued.

$= \frac{1}{4}(1\cdot 24)$	Reunitize the 3 groups of 8 to 1 group of 24.
$= \frac{1}{4}(4\cdot 6)$	Partition and reunitize to 4 groups.
$= 1\cdot 6$	Replace the 4 groups of 6 with 1 group of 6.
$= 6$	Reunitize 1(6-unit) to 6(1-unit)s.
c. $\frac{3}{4}(8) = \frac{3}{1}\left(\frac{1}{4}(1\cdot 8)\right)$	3/4 expressed as composite of 3/1 and 1/4 and 8 groups of 1 as 1 group of 8.
$= \frac{3}{1}\left(\frac{1}{4}(4\cdot 2)\right)$	Replace 1 group of 8 with 4 groups of 2.
$= \frac{3}{1}(1\cdot 2)$	Exchange 4 groups of 2 with 1 group of 2.
$= 3\cdot 2$	Exchange 1 group of 2 with 3 groups of 2.
$= 6$	Exchange 3 groups of 2 with 6 groups of 1.

Curricular implications. This analysis of rational number as a D/PR exchange operator suggests several prerequisite knowledge structures: (a) ability to partition quantities, (b) flexibility in formation and re-formation of units, (c) understanding of and ability to perform partitive division, (d) an understanding of the concept of function as a mapping, and (e) skill with and understanding of multiplication as repeated addition.

Application of the D/PR construct to problem solving.

A major issue in developing problem-solving skill rests with the ability of students to form a representation that accurately reflects the quantities in the problem and the relationships among these quantities. Two matters concerning word problems relate to so-called extraneous-data and multiple-step problems, both make problems difficult for children to solve. To illustrate, consider the following problem situation and two questions.

Problem Situation. Many brands of gum are sold in the form of packages with 5 sticks in a package. Jane has 8 packages of gum. Mary has 3/4 times as much gum as Jane. Question 1. How many packages of gum does Mary have? Question 2. How many sticks of gum does Mary have?

In traditional problem-solving instruction, the information that there are 5 sticks in each package would be considered extraneous data for Question 1 because this question could be answered without that information. Nevertheless, the presence of this data causes difficulty for children. One reason for this might be that the model that is used in traditional instruction to answer Question 1 is not an accurate model of the problem situation. That there are 5 sticks in each package is part of the situation. Would problem-solving performance be improved if symbolic models to answer Question 1 could more accurately model the situation? When concern is for an answer to Question 2, traditional instruction classifies the problem as a multistep problem. A difficulty for children in solving multistep problems is that carrying out the first step (in this case multiplying 8 times 5 or 3/4 times 8) introduces still another quantity into the situation and the relationship of this new quantity to the existing quantities must be established

We will interpret the problem situation in the generalized mathematics of quantity notation and show that the same initial representation can be used to answer both questions. Differences in the solution process will be seen to depend on a different re-formation of units of quantity. The quantities are as follows: each 5 sticks of gum is 1(5-unit); 8 packages of 5 sticks, a unit-of-units, is 1(8(5-unit)s-unit), which is also 1(8-unit); 3/4 is initially taken in the generic sense of a 3 (1-unit)s for 4 (1-unit)s exchange, as the solution progresses we think of it as a a 3 (2(5-unit)s-unit)s for 4 (2(5-unit)s-unit)s exchange (See Steps 2 and 3.). The important issue for 3/4 as a D/PR is that 4 units are exchanged for 3 units of the same type.

Figure 4. A mathematics of quantity representation: problem and solutions.

1.	$\frac{3(1\text{-unit})s}{4(1\text{-unit})s}$ 1(8(5-unit)s-unit)	The model of the situation.
2.	$= \frac{3(1\text{-unit})s}{4(1\text{-unit})s}$ 4(2(5-unit)s-unit)s	The (8-unit) is reunitized to 4(2-unit)s.
3.	$= \frac{3(2(5\text{-unit})s\text{-unit})s}{4(2(5\text{-unit})s\text{-unit})s}$ 4(2(5-unit)s-unit)s	The conception of 3/4 as a 3-for-4 exchange function is changed to fit the context of the situation.
4.	= 3(2(5-unit)s-unit)s	The function rule is applied, 4(2-unit)s is replaced by 3(2-unit)s.
5.	= 1(6(5-unit)s-unit)	A reunitization.
6.	= 6(5-unit)s	A reunitization.

(Given the initial interpretation that a package of gum is a (5-unit) we have the answer to Question 1. Mary has 6 packages of gum.)

7.	= 30(1-unit)s	A reunitization.

(Making the interpretation that each stick of gum is 1 (1-unit), we have the answer to Question 2. Mary has 30 sticks of gum.)

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CHILDREN'S USE OF OUTSIDE-SCHOOL KNOWLEDGE TO SOLVE MATHEMATICS PROBLEMS IN-SCHOOL

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In this paper we report on a research project carried out among Portuguese children, the aim of which was to investigate the relationship between outside and in-school knowledge. Data from case studies are presented in terms of (1) the kind of knowledge children bring with them (2) the influence of socio-cultural constraints on their uses of mathematics.

In recent years there has been a considerable number of studies conducted on research in mathematical cognition relating to individual competences in contexts of everyday life (eg. Carraher, 1988; Lave, 1988; Saxe, 1990). The results of those studies, together with the development of the awareness in the field of mathematics education that mathematics is a form of cultural knowledge (Bishop 1988, D'Ambrosio, 1985), draw attention to the gap between the use of mathematics inside and outside school. However, although this gap is evident, as is the idea of the need to bridge it, there is not enough research which clarifies how to operationalize that proposal in the curriculum. Carraher (1988) in her plenary address at the PME XII conference stressed this point when she said "building bridges between street and school mathematics appears to be a route worth investigating in education" (p. 19).

In this paper we will report the first stage of an ongoing research project where the central aim is to investigate the nature of the relationship between the mathematical knowledge acquired by children in their out-of-school culture and their performance in school mathematics. Specifically we are interested in two questions: what kind of knowledge do children bring with them into school, and what are the socio-cultural constraints which allow, or prevent, the use of outside knowledge in school!

In relation to the first question, what the school aims to teach to children is clearly established in a mathematics curriculum, but what the child will bring from their outside school life cannot possibly be standardized. As Saxe (1990) stated, the level and kind of understanding of out-of-school mathematics practice is variable and will depend on the level and kind of experience and participation by the children in that practice. The same aspect has been documented by Abreu (1988) in a study where she found that although all the sugar cane farmers shared a specific body of

measures for length and area, they performed area calculations at different levels, according to their routine practices. These findings suggest that the kind of outside school knowledge children can bring could be different from one child to the other, depending on the nature of the specific social practices they experience in their out-of-school contexts.

In relation to the second question, Harré (1986: 294) states: "developing human beings change not only in respect to what they know they can do, but also and most importantly to what their society permits them to do". This quotation, when translated into the school situation, can be rephrased as: children at school, as well as learning the knowledge, also acquire norms of 'what the socio-cultural constraints of school permits them to do'. So, a child who has the competence to solve mathematical problems in an 'outside' situation could be unsuccessful at school, because s/he believes that that knowledge is not suitable for use in school, or vice-versa.

The research project

The context of this investigation was certain primary schools in the rural parts of the Madeira island in Portugal. To improve the quality of schooling is one of the first items in Portuguese EC agenda as the country shows high rates of grade repetition (37% in the two first grades of the primary schools in Madeira). For this study we chose the Camara de Lobos council, an area of about 52 Km², with 33,500 inhabitants, and an economy based on farming and fishing. Similar to the situation of the Brazilian children studied by Carraher, those children of Camara de Lobos who are failing in school are generally heavily involved in outside school activities to help the families, mainly in farming.

The participants were the pupils of three third grade classes and one sixth grade class in two primary schools. From this group, 12 children were selected as case studies, balancing boys with girls, and also children with difficulties in mathematics at school with children who are successful, according to the teachers' judgements.

A four-stage research approach was adopted. The first stage involved classroom observation. The second was an individual interview about the child's life outside school and his/her beliefs about mathematics. The third stage involved group activities where the children were presented with two kinds of tasks involving measurement: (a) tasks where they were asked to imagine they were farmers; (b) typical school tasks, taken from their textbooks. The fourth stage was a final individual interview and again they were asked to solve tasks about measurement. All the stages were carried out by the same researcher in the two schools. The interviews were audiotape recorded, written notes were taken during the interviews, and the children's written calculations were also kept.

The results

There is no room here to present and discuss all the results, and so we will only quote some of them, together with extracts from interviews, which relate to the two initial questions (more data will be presented at the conference).

(1) What kind of knowledge do they bring?

Children bring with them various kinds of mathematical Knowledge, among which we documented: (a) knowledge about out-of-school situations where people use mathematics; (b) Knowledge about the social practices in which they engage and the 'emergent mathematical goals' (Saxe, 1990) in those situations; (c) knowledge about specific mathematical concepts, such as the measurement of length and area. Each of these will now be described.

(a) Out-of-school situations where people use mathematics

Table 1 sumarizes children's beliefs about some out-of-school situations where people need to use mathematics. We presented different pictures and asked children about the need to use mathematics in each situation.

Table 1: Children's beliefs about the need to use mathematics in out-of-school situations and their justifications

pictures	frequency	justifications
1) public fish market	11 (92%)	. To count money; to do sums; to cope with change
2) carpenters making furniture	10 (83%)	. to measure; to communicate to the customer; to do sums to earn money; to sell but not to make furniture.
3) children measuring and drawing a circle	7 (58%)	. to measure; to draw a circle and angles; to count; to work to earn money.
4) people working in farming	8 (66%)	.to sell their products; to distribute the seeds; for the worker to receive the money; to count money.
5) people in a supermarket	12 (100%)	. to buy and sell.

We found that for the commercial activities of selling or buying, for example in a public fish market and in a supermarket, all the children but one, agreed about the kind of mathematics needing to be used, that is doing 'sums', counting money, checking change. However, for the other three

situations we found that some of the children could not see any mathematics there, and that those who did gave a variety of different justifications. For example, in the carpentry and farming picture some children could identify the mathematics only where it was linked to the stage of selling the product. In general, the children's beliefs seem to be linked to their personal out-of-school practices. One exception was found in relation to picture three, where only the sixth grade pupils could identify circles and angles as part of mathematics. The level of schooling thus seems to have an influence in broadening the children's beliefs about what is mathematics, even though we found no differences related to children's performance at school.

b) Knowledge about the social practices they engage in and the 'emergent mathematical goals' of those situations

By asking the children to report on the activities in which they normally engaged in order to help their families, we found that the majority of children helped their mothers in housework, which for some included buying food in the nearby groceries. All the boys engaged in farming activities, some of them helping their parents and others working in the neighbourhood to earn money. Some of the girls also helped in farming, but in different tasks from the boys. We also asked them to describe some of the routine activities they do in order to verify the 'emergent mathematical goals' in those situations. We have chosen to analyse here how two girls explain the activity of buying bread for the family. The structure of the activity can be described in three phases: preparing to buy bread; buying bread ; checking the result at home. Now let us see how the two girls describe the activity:

Example 1: Rita, a fisherman's daughter, 10 years old, in the third grade, described by the teacher as having difficulties at school:

Rita - Saturday morning is my day, I go to fetch bread early in the morning.
I - And how many loaves of bread do you fetch?
Rita - Fifteen.
I - Who gives you the money?
Rita - My mother, my father and when they don't have any money I do not go.
I - And how much does each loaf cost?
Rita - 30 (escudos)
I - And how do you know, for example, how much seven loaves cost? Who does the sums?
Rita - The seller. But, if I arrive home and the sum is wrong I need to go back.
I - But, when you arrive home who checks if the sum is wrong?
Rita - Me [she demonstrates writing an addition where for each loaf she writes 30 escudos].

***Example 2**: Ana, a farmer's daughter, 8 years old, in the third grade, described by the teacher as a successful pupil:*

I - [she said that she used to buy bread for the family, she used to buy 3 loaves and they cost 90 escudos] . And if you bought five how much would they cost?
Ana - I do not know, we never buy five.
I - And if your mother asks you to buy only one, how much will you pay?
Ana - One I do not know. My mother gives me the money and if it is short he (the seller) tells me and the next day I bring the money that was missing. If it is not short he gives me the change.

Looking at the two examples we verify that the structure of the activity is the same for the two children, although the kinds of social constraint imposed by the interactions in the family seem to be rather different. Rita copes with a situation where she needs to use mathematics. When buying she needs to make sure she gives the right amount to the seller or receives the correct change, and when checking at home to see if the change is correct. On the other hand, Ana does not need to solve any mathematical problems during the three phases, since her family solves them for her. The point that we want to stress here is that the 'emergence of a mathematical problem' in the out-of-school world, although children can be involved in similar practices, seems to be linked to the social constraints they experience. Moreover, although Ana has the best performance in mathematics among the girls in school, she does not seem to link any mathematics to that outside situation. In contrast, Rita, the pupil with difficulties, can cope mathematically with the situation, although she uses very simple strategies, probably the ones she feels more confident with.

c) knowledge about specific mathematical concepts, such as the measurement of length and area

In a group activity, the children were asked 'to imagine they were farmers' and to solve a sequence of length and area measurement tasks. Their answers showed their understandings about: perimeter and area, conservation of length and area; how to measure; how to use a ruler; the inverse relation between the size of unit and the number resulting in the measurement; and how they cope with halves. Here we will refer to this last point. The problem was created by giving the children a wooden stick and asking them to use it to calculate the amount of wire they needed to fence all around the farms. The 'farms' were straight-sided shapes, the stick was 6 cm long, but it did not fit neatly into the length of all the sides. The different strategies children used to cope with that task are described below:

(1) Iteration around the perimeter of the farm, considering the halves, but counting on only the integers until they find another half and then compensate (used by four third grade girls, two successful and two with maths difficulties).

(2) Iteration around the perimeter of the farm, considering the halves, and incorporating them in the counting on (used by three third grade boys, one successful and two with difficulties).

(3) Iteration on each side, considering the halves, and subsequently adding the sides, without the need to measure the parallel sides in the rectangle (used by one successful third grade boy, and by three sixth grade pupils, one successful boy and two with difficulties: a boy and a girl)

(4) Iteration around the figure, turning the stick at the corners, counting on the integers with no need to refer to the halves (one sixth grade successful girl).

All the third grade children, independent of their actual school performance, brought understandings about adding two halves and then completing one unit, although they had not yet been introduced to fractions or decimals at school. Concerning the sixth grade pupils, they did the task very well when using 'mental calculations', but when they decided to use the school algorithm for adding decimals they failed. The most worrying point was that they did not realize their mistakes, although the result could be a number 'smaller' than only one side.

(2) What are the socio-cultural constraints?

Outside school the children could be free to choose the strategies about which they felt more confident. But, in school they had to conform to the socio-cultural constraints imposed by the system. We found that: (a) pupils when confronted with a school task follow a school procedure to solve it, although they can do that more accurately using another procedure; (b) pupils believe that school mathematics methods are in some way superior to the out-of-school methods.

(a) Solving a problem taken from their maths textbook:

> The ribbon measures 21 meters. If we use 12 meters of ribbon, how much will be left?

Looking at the solutions of the third grade pupils we verify that all but one adopted the following procedure: reading the problem; discussing what operation is required; concluding that it is a subtraction or what they call a 'minus sum'; and performing the written calculation. All of them obtained a wrong result! At this point they were asked to do the same calculation without writing, and were explicitly allowed to count on their fingers. Again, without exception, all the children succeeded in finding the correct answer. Now we will quote a unique interview where the pupils inverted the usual process.

Joel - [after reading twice] Nine is left.
I - Your partner is saying that nine is left, how would you say that answer to your teacher?
Nuno - The ribbon has nine meters.
Mario - It cannot be like that.
I - Have a go.
Mario - The ribbon has nine meters left.
I - Now that sum that you did in your mind, do it in writing.
Joel - Do it this way, put 21 minus 12 [numerical expression]. Now put 21 and below 12, after the minus sign, now do the sum, the result is nine.
Mario - I know how to do this. Two minus one is one, and two minus one is one. Oh! No. Eleven! [he checks the answer again]
Joel - [does the calculation again with his fingers] It is nine. Count [demands to the partner] from 12 to 21.
Nuno and Mario - [both count] It is nine.

The spontaneous discussions and checking of the answer only emerged in this group, which began with the mental calculation. In the other groups they accepted the wrong result without any doubt, although in some cases they obtained a number bigger than the 'subtrahend'. This result suggests that the kinds of constraint which school is imposing could negatively affect children understandings and progress in mathematics.

(b) Children's beliefs about outside and school strategies

Children can identify differences in the way they or their parents solve a problem outside and the way they do it in school, but generally they believe that school strategies have advantages. Some also believe that the outside strategy is not adequate, as in the next example:

I - And do you do the sums at home in the way you have learned at school?
Rita - Some.
I - And the others, how do you do them?
Rita - 'wrongly' (ie. not the proper way).
I - Is it 'wrongly' or you do it in a different way?
Rita - In a different way.

Besides the children's explicit references to the adequacy of school strategies and to the certainty they felt about the result they obtained when solving problems using school methods, another aspect which we observed is that a great number of pupils internalized the school addition algorithm to solve mental additions. Perhaps this tendency is associated with their beliefs about the superiority of school methods.

Conclusion

It is clear from results like these that children bring with them a variety of outside school mathematics understanding, involving both specific knowledge and beliefs. In particular, considering the five sets of results presented here, we can see that:

(a) Children can identify some of the mathematics that people need to use in several different out-of-school situations. Their beliefs about the kinds of mathematics people need to use outside are very close to their beliefs about the importance of learning mathematics at school. Their beliefs seem not to be related to their performance at school, but rather they seem to change with the advance of schooling.

(b) Children cope with a variety of out-of-school social practices, in which they use meaningful mathematics when demanded.

(c) Children have prior understandings about specific mathematical concepts, such as dealing with halves in measurement, which are more meaningful than when they learn the school methods.

(d) Children when confronted with school-tasks tend to use school methods and do not doubt the results. However they demonstrate that they are more accurate when solving with their own strategies.

(e) Children have the belief that school methods are more suitable than their out-of-school methods.

It is evident that when allowed, children can make explicit their outside school knowledge in a school context of research. This seems to be a starting point to make the bridge. It is also suggested that the outside school methods could be the beginning of a more meaningful school mathematics. However, some questions still remain: How will this process work as a routine inside real classrooms? To what extent can pupils be enabled to make this link between outside and in school mathematics, and will this improve their learning at school?

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INFLUENCES OF AN ETHNOMATHEMATICAL APPROACH ON TEACHER ATTITUDES TO MATHEMATICS EDUCATION

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ABSTRACT

This report describes a recent research study carried out in Brazil concerning the influences of an ethnomathematical teaching approach on teacher attitudes to mathematics education. Some of the data collected will be presented, and conclusions drawn.

INTRODUCTION

One of the biggest problems facing the Brazilian educational system today is the number of early school leavers. According to the Brazilian Constitution it is compulsory for all children between seven and fourteen years to attend school. However, from each hundred children registered every year in the first school year, only fifty-one of them will be registered in the third school year (see B.S.Y., 1986). There are several reasons for this kind of problem, the main one being the economic situation of the ordinary Brazilian family. Many children have to leave school because they need to work to earn money for their families. Another reason relates to the teaching of school subjects, and here we are particularly concerned about the quality of school mathematics. The fundamental question we are trying to investigate is: "Where is school mathematics failing?".

One recent development in thinking about school mathematics concerns the idea of ethnomathematics. As defined by D'Ambrosio [1985], "ethnomathematics is the mathematics practised among identifiable cultural groups, such as national-tribal societies, labour groups, children of a certain age bracket, professional classes, and so on." (p.45)

So far some theoretical positions have been put forward, but few practical applications have been developed and researched. A recent study has started in Brazil which is concerned with understanding the process of bringing ethnomathematics into the classroom, and which focuses on the teachers' role in this micro-curriculum development process. In this report we will outline the research study and report some of the findings which relate to the teachers' attitudes towards mathematics education.

The main research questions asked were:

(1) What are the attitudes of teachers to mathematics education in relation to ethnomathematics and the traditional Brazilian approach to mathematics teaching?

(2) How strongly do teachers agree about these attitudes?

(3) What changes in attitudes take place during an experimental application of an ethnomathematical approach?

THE RESEARCH STUDY

The procedure used in this study can be described in four different phases:

PHASE I: THE THEORETICAL WORK WITH TEACHERS

The aim of this phase was to characterize an ethno - mathematical approach for the teachers, and to identify the different human activities which could be mathematically interpreted in the classroom. Classes about these activities were organised and conducted for the nineteen teachers who were taking part in the research.

For characterizing such an ethnomathematical approach, the model of analysing a curriculum suggested by Robitaille and Dirks (1982) was adopted. According to their analysis, a mathematics curriculum exists at three different levels: "We may distinguish among the curriculum as intended, the curriculum as implemented, and the curriculum as attained." (p.17) This three level model was applied:

(a) to the five different theoretical approaches to mathematics curriculum suggested by Howson, Keitel and Kilpatrick's (1981) work in the area of curriculum development:

(a.1) The Behaviourist Approach,

(a.2) The New-Math Approach,

(a.3) The Structuralist Approach,

(a.4) The Formative Approach,

(a.5) The Integrated-teaching Approach; and

(b) to the ideas of Bishop (1988), D'Ambrosio (1985) Gerdes (1988), Mellin-Olsen (1987) and Zaslavsky (1987) in which mathematics is seen as a 'Cultural' phenomenon, a sixth theoretical approach.

The result of these applications of the Robitaille and Dirks' model, together with an analysis of the "Curriculum Proposal for the Teaching of Mathematics - First Grade" of the State of Sao Paulo in Brazil, was a theoretical characterization of an "Ethnomathematical" approach in comparison with what we called a "Canonical - Structuralist" approach. This theoretical characterization, contrasting the experimental 'ethnomathematical' approach with the Brazilian official approach, the 'canonical structuralist' approach, was the foundation of the research study. In particular, it determined the form of the questionnaire instruments used to monitor the participants' attitudes.

The two approaches are summarized in Table 1, below:

THE CANONICAL-STRUCTURALIST APPROACH	THE ETHNOMATHEMATICAL APPROACH

A GENERAL PERSPECTIVE

Mathematics should be seen as:

a) a THEORETICAL subject (it concerns abstractions and generalisations);	b) a PRACTICAL subject (it is applicable and useful);
c) a LOGICAL subject (it develops internally consistent structures);	d) an EXPLORATORY and EXPLANATORY subject (it investigates environmental situations);
e) a UNIVERSAL subject (it is based on universal truths).	f) a PARTICULAR subject (it is based on truths derived by a person or group of persons).

AT THE INTENDED CURRICULUM LEVEL

The mathematics curriculum should:

a) be CULTURE FREE (its truths are absolute, and independent of any kind of cultural or social factors);	b) be SOCIALLY/CULTURALLY BASED (its truths are relative, and dependent on social or cultural factors);
c) be INFORMATIVE (it emphasizes procedures, methods, skills, rules, facts, algorithm and results);	d) be FORMATIVE (it emphasizes analysis, synthesis, thinking, a critical stance, understanding and usefulness);
e) be CONSERVATIVE (it promotes control over the environment and the stability of the society).	f) be PROGRESSIVE (it promotes the growth of knowledge about the environment and progress/change of the society.

AT THE IMPLEMENTED CURRICULUM LEVEL

Teachers should teach mathematics as:

a) a ONE-WAY SUBJECT (mathematical knowledge is transmitted from the teacher to the pupils);	b) a DEBATABLE SUBJECT (mathematical knowledge is discussed among pupils and teachers);
c) a SEPARATED SUBJECT (mathematical lessons do not rely on knowledge which pupils bring from outside school);	d) a COMPLEMENTARY SUBJECT (mathematics lessons are based on knowledge which pupils bring from outside school);
e) a REPRODUCTIVE SUBJECT (mathematical knowledge is taught from standard mathematical texts).	f) a PRODUCTIVE SUBJECT (mathematical knowledge is developed from the pupils' own situations).

AT THE ATTAINED CURRICULUM LEVEL

Pupils should be able to:

a) FIND CORRECT ANSWERS to problems (it is the pupils' final answers to problems which are important);	b) ANALYSE PROBLEMS (it is the pupils' understanding of the structure of a problem which is important);
c) USE THE FORMAL MATHEMATICAL METHOD to solve problems (these methods are the ones which will produce the right solutions);	d) USE APPROPRIATE PROCEDURES to solve problems (it is the pupils' ability to determine the appropriate solution procedure which is important);
e) REASON MATHEMATICALLY about problems (it is HOW to solve problems mathematically that is important for pupils to know).	f) MAKE MATHEMATICAL CRITICISM about problems (it is WHY to solve problems mathematically that is important for pupils to know).

PHASE II: THE PRACTICAL WORK WITH TEACHERS

In this phase the teachers and the researcher planned and developed six different teaching projects based on the characterization of 'Ethnomathematics' and on the human activities identified in the first phase. Each project was designed as a micro-curriculum sequence of activities based on the pupils' knowledge of a particular aspect of their reality. The teaching projects developed were the following ones:

(a) "The Skipping Rope" project - a *counting* project based on activities involving skipping games with ropes, by pupils of between 6 - 10 years;

(b) "The Hop Scotch" project - a *locating* project exploring the layout and organisation of different 'hop scotch' patterns with pupils of 9 - 13 years;

(c) "The Windmill" project - a *measuring* project where different geometrical figures are used to make windmills with pupils of 9 - 15 years;

(d) "The Queimada Game" project - a *playing* project based on a popupar children's team game, where pupils of 10 - 15 years experimental with the shape of the playing area and the rules of the game;

(e) "The Hot Air Paper Balloon" project - a *designing* project where pupils of 11 - 16 years designed and made different balloons having been give certain dimensions;

(f) "The Brazilian Economic Plan" project - an *explaining* project where pupils of 13 - 16 years studied and explored the meanings and implications of different aspects of the plan.

PHASE III: THE APPLICATION OF THE SIX TEACHING PROJECTS

In this phase the 19 teachers applied the teaching projects in eighteen different classes of twelve ordinary council/state Brazilian schools. These projects were applied to about four hundred and fifty (450) pupils, and each project was used for between 3 and 5 weeks (i.e. between 15 - 25 hours of mathematics class time).

(A video of some of the applications of the teaching projects will be presented at the conference)

PHASE IV: THE ANALYSIS OF THE TEACHING PROJECTS AND THE RESEARCH STUDY AS A WHOLE

The aim here was to study and analyse the effects of the teaching projects, from the point of view of the teachers, the pupils and the parents. This aim was reached through the application of a series of questionnaires to teachers, pupils and parents and through observation and interviews with the teachers.

In particular, one questionnaire was given to the teachers on three separate occasions during the research: before and after the first phase, and after the third phase. The objective of this was to investigate changes in teachers' attitudes concerning different aspects of the ethnomathematical approach. This questionnaire was closely based on the analysis given in Table 1. Thus, this questionnaire was designed and applied as a kind of 'thermometer' to register, at specific points in the research project, the thinking and the reactions of the teachers about the ethnomathematical teaching methodology which was being presented and developed with them.

SOME DATA FROM THE TEACHERS' QUESTIONNAIRE

The questionnaire had four questions, reflecting the four sections of Table 1, each one with six alternatives labelled as in Table 1. The teachers had to rank these alternatives according to their own opinions about the question being asked. In addition, for each question, a space was given for teachers' comments or personal definitions.

Applying the "Kendall Coefficient of Concordance (W)" [Siegel, 1956] to the answers given to those questions revealed that the value of 'W' (i.e. the extent of teachers' agreement) was significant at the .01 level for all the questions and for all its applications. Therefore, it is possible to define a "Standard Teachers' Answer" for each of the four questions and in each of the three applications. TABLE 2 below shows the standard answers obtained for each question on the questionnaire:

TABLE 2

STANDARD TEACHERS' ANSWERS

Question	1st Aplication						2nd Aplication						3rd Aplication					
	1st	2nd	3rd	4th	5th	6th	1st	2nd	3rd	4th	5th	6th	1st	2nd	3rd	4th	5th	6th
1	b	c	d	e	a	f	b	d	c*	f*	e	a	d	b	c	f	e	a
2	d	c	b	f	e	a	d	f	b	a	c	e	d	f	b	c	a	e
3	f	b	d	e	a	c	b	d	f	e	c	a	b	f	d	e	c	a
4	b	f	e	d	c	a	b	f	d	e	a	c	b	d	f	e	c	a

Where: - the meaning for each alternative response (a, b, ... f) can be seen in TABLE 1;
- '*' means that the alternatives were ranked in equal positions.

Looking at such standard answers, for *question 1* on the general perspective (For me, mathematics is ...), an important remark is the fact that alternatives 'b', 'c' and 'd' were ranked in the first three positions on each occasion. It means that mathematics, for those teachers, was seen basically as a 'practical', 'logical' and 'exploratory and explanatory' subject. In addition, alternatives 'd' and 'f' were the only one which increased in importance throughout the study.

A teacher's written statement which amplifies such findings is: "In this research study, I acted in two different, but possible and complementary ways: as a teacher and as a learner. As a teacher, I tried to show to my pupils a new face of mathematics. A face which is hidden by the school. A mathematics which can be seen as practical, pleasurable, formative and necessary. As a learner, I changed my own view about this subject with the aim of reaching a 'reconciliation' which I judged to be impossible before."

Analysing the standard answers of *question 2* on the intended level (For me, mathematics occupies an important place in the school curriculum because ...), there are three important remarks which should be made. First, the fact that alternatives 'b' and 'd' (mathematics is 'socially/culturally based' and 'a formative subject') were the only ones which did not change their rank on the three occasions that the question was answered (the 3rd and the 1st positions respectively). Second, alternative 'f' (mathematics is a 'progressive subject') showed the biggest increase in importance in the rank throughout the research study. Finally, the alternative 'c' (mathematics is an 'informative subject') showed the biggest decrease in importance in the rank during the development of the research study.

Analysing the standard answers of *question* 3 on the implemented level (For me, mathematics should be taught as ...), the main remarks are as follows. The alternatives 'b', 'd' and 'f' (a 'debatable', a 'complementary' and a 'productive' subject) occupied the first three rank positions on each of the three applications. On the other hand, alternatives 'a' and 'c' (a "one-way" and a 'reproductive' subject) occupied the two lowest rank positions during the whole of the research study.

A teacher made the following remark about this question: "I believe that the way in which mathematics is normally taught today, it only contributes to minimizing the number of people who would like it. The way of teaching it should be clearly linked to the teacher's attitude in front of his/her pupils. The majority of teachers think that it is easier, and more convenient, to 'repeat' mathematical concepts, assuming that they are absolute truths. In such a way there can be no discussion, and there is no space for pupils to question, to find out, to analyse."

Finally, analysing the standard answers of *question* 4 at the attained level (For me, pupils who have learnt mathematics should be able to ...), it is important to emphasise that alternative 'b' ('analyse problems') was ranked in the first position in all the three applications. In addition, alternative 'd' (use 'appropriate procedures' to solve problems) showed the biggest increase in importance during the research study, while alternatives 'e' and 'f' ('reason mathematically' and 'make mathematical criticism' about problems) decreased in importance. Also alternatives 'a' and 'c' ('find correct answers to problems' and 'use the Formal mathematical methods to solve problems') were those which were ranked in the two lowest rank positions throughout the research study.

(Further data from the study will be available at the conference.)

CONCLUSION

Overall, at the end of the whole research experience, the teachers' prefered views were that:

(1) Mathematics should be seen as an *Exploratory* and *Explanatory* subject, investigating environmental situations.

(2) The mathematics curriculum should be *Formative*, emphasising analysis, synthesis, thinking, a critical stance, understanding and usefulness.

(3) Teachers should teach mathematics as a *Debatable* subject where mathematical knowledge is discussed among pupils and teachers.

(4) Pupils should be able to *Analyse Problems*, and understand the structure of problems.

Although the Standard Teachers' Answers used in this report cannot characterize completely the extent of the changes which took place in teachers' attitudes to mathematics teaching, it is of interest to remark that after the research study these teachers became much more concerned with many of the aspects emphasised by the ethnomathematical approach at its four levels of analysis.

However, one question which remains is whether a simple shift from one characteristic to another, or from one approach to another, is enough to guarantee better results in mathematics teaching? In fact, what we will need to do is to determine the appropriate balance between the characteristics stressed by these two approaches. We (educators) should necessarily take into account both views of mathematics education, not just one of them as usually happens today. We should be concerned not only about the macro-dimension of the mathematics curriculum,but also with the micro-dimension of it. We should look more carefully at school mathematics from the pupils' point of view with the aim of determing what kind of "knowledge environment" (Bishop, 1988) we can most appropriately create for them.

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Gender and the versatile learning of trigonometry using computer software

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This empirical study tests the hypothesis that the versatile learning of trigonometry using interactive computer graphics will lead to a greater improvement in the performance of girls over boys. The experiment was carried out with 15 year old pupils in two parallel halls of a comprehensive school, each subdivided by ability into four corresponding mixed gender groups. In every case, experimental boys improved more than control boys and experimental girls improved more than control girls. However, whilst the control boys improved more than the control girls, the experimental girls improved more than the experimental boys, eventually becoming superior in all but the least able group.

Difficulties in the learning of trigonometry

The initial stages of learning the ideas of trigonometry are fraught with difficulty, requiring the learner to relate pictures of triangles to numerical relationships, to cope with ratios such as $\sin A = opposite/hypotenuse$ and to manipulate the symbols involved in such relationships. Ratios prove to be extremely difficult for children to comprehend (Hart 1981), and modern texts have responded to the perceived difficulties by introducing the sine of an angle not as a ratio, but as the opposite side length in a right-angled triangle with unit hypotenuse which must be recognized with the triangle rotated into any position.

Further difficulties occur as the student must conceptualize what happens as the right-angled triangle changes size in two essentially different geometric and dynamic ways:

- as an acute angle in the triangle is increased and the hypotenuse remains fixed, so the opposite side increases and the adjacent side decreases,
- as the angles remain constant, the enlargement of the hypotenuse by a given factor changes the other two sides by the same factor.

The traditional approach uses pictures in two different ways, each of which has its drawbacks. Rough sketches of triangles may give the impression that the numerical procedures are the only way to get accurate results, downgrading the role of pictures in the minds of the students. On the other hand, if students draw an accurate diagram, this focuses on the production of one static picture rather than the visualization of dynamically changing relationships.

A computer approach can change all this by allowing the child to manipulate the picture and relate its dynamically changing state to the corresponding numerical concepts. It therefore has the potential of improving understanding. This ability to use the computer to carry out certain arduous constructions whilst the child can focus on specific relationships we call ***the principle of selective construction***. We believe this to be one of the most powerful educational principles for the use of the new technology.

Gender differences in mathematical performance

Empirical evidence shows that although girls perform at least as well as boys in mathematics in the early years, as they get older a divergence in performance may become apparent, particularly amongst the higher ability groups. For example, the percentage of boys and girls obtaining the highest grades (A,B,C) in the U.K. examinations taken at the age of sixteen (the General Certificate of Secondary Education) is as follows:

	% obtaining grade A		% obtaining grades A,B,C	
	1985	1987	1985	1987
Boys	9.4%	9.1%	41.2%	40.5%
Girls	4.8%	5.2%	33.7%	33.5%

(Source: D.E.S. 1989)

table 1

There is also evidence that girls of this age are less successful at visuo-spatial tasks than boys, particularly in the highest ability groups. We hypothesised that if software was designed to link together spatial concepts and numeric and symbolic data, then this may aid girls in perceiving linkages where the traditional UK curriculum leaves them with a perceived deficiency. In addition, observed differences in social behaviour in earlier experiments (Tall & Thomas 1988), in which girls were seen to be more likely to cooperate, whilst boys would often compete, might enhance the girls' success rate in gaining conceptual insight.

Software designed to improve versatile learning of trigonometry

To improve versatility in relating numerical and visual cues a simple piece of software was designed linking numerical input for lengths and angles to a visual display of a right-angled triangle ABC. Any of the sides and angles can be specified and, once sufficient information is available, say two sides, or a side and an angle (in addition to the fixed right angle), the triangle will be drawn (figure 1). The user may then choose to reveal any other sides or angles and manipulate the data in various ways, for instance to add, subtract, multiply or divide any length or angle by a given quantity. The triangle may be rotated and drawn either to a fixed scale or auto-scaled so that it remains the same size onscreen when all sides are adjusted by the same factor.

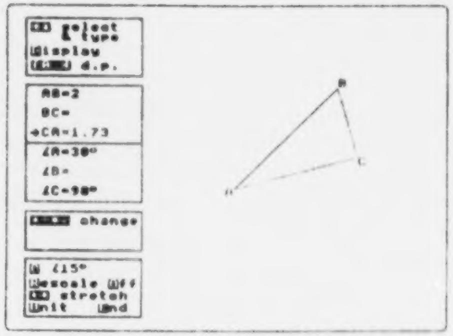


Figure 1 : the screen display of the trigonometric software

By showing *examples* of trigonometric phenomena relating numeric and geometric representations, the software acts as a *generic organizer* in the sense of Tall (1986), allowing the teacher to demonstrate, and the pupils to explore, examples of the concepts being introduced. The teacher needs to act as an *organizing agent*, focussing the pupil on the important ideas, then encouraging pupil interaction with the software to enable them to make personal constructs of the interrelationship of the mathematical ideas. By using the principle of selective construction to focus on what are normally considered to be higher order relationships it was hypothesised that the students would gain more versatile understanding of trigonometric concepts.

Experimental design

The experiment was carried out in Kenilworth School which teaches the full range of ability and is organized as two parallel halls for pupils in the age range 12-16. In each hall the children aged 14-15 are streamed into four mixed gender classes according to ability. The hall which had performed marginally better on the previous year's examination was selected as the control, with the classes taught by the normal class teachers, the other was given the experimental treatment by one of the authors (a former teacher at the school). The experimental teacher, travelling some distance to the school, had little time to do any more than arrive for his class, teach, then leave again, and any advantages, with the exception of the use of the computer, tended to accrue to the control groups.

A pre-test given to all but the least able confirmed that there was no significant difference between the groups being compared (see tables 5-8 below). All groups had 4 lessons of 1 hour 10 minutes and 4 lessons of 35 minutes in a two week period, followed by an immediate post-test and a second post-test eight weeks later.

After the first post-test the teachers of control groups 1 and 2 chose to discuss their student's mistakes and give them extra practice before the second post-test. This could be construed as giving these pupils an advantage over the experimental students but, as we will see, it did not improve their relative performance.

Aims in teaching trigonometry with the computer

The computer representation enabled the students to explore the relationship between numerical and geometric data in an interactive manner. For instance, at an early stage they used the computer to start with an angle of 10° and to tabulate the changing values of the opposite and adjacent sides as the angle is increased in steps of 10°, giving them early insight into the complementary relationship between the increasing table of sines and the decreasing table of cosines. From the outset they were encouraged to make dynamic links between visual and numerical data which is less apparent in a traditional approach.

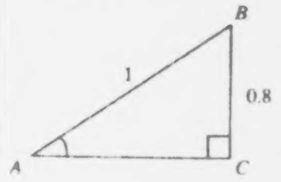
Over the two week period the experimental lessons developed in a versatile manner: using the computer to estimate numerical values before checking them with the computer and with a hand-calculator : estimating sides for given angles and angles for given sides, scaling triangles up or down in size, solving related problems. Three computers were available for group use, with care taken to give the girls their fair share. This was only a problem in experimental group two where the boys competed to try to monopolise the computer.

Students of all abilities found the experimental approach reasonably straightforward, with the less able encountering problems for instance in handling decimals less than one. Even the least able became adept at using the computer and, though they had some difficulty writing down their results, they had few difficulties with visualization. This has obvious implications for the rigidity of differentiated curriculum schemes, where it is decided that certain conceptual structures are too difficult to teach certain children.

Differences in responses of experimental and control pupils

The differences in conceptual development are shown most clearly on the delayed post-test which consisted of 5 problems to test more standard trigonometric techniques and 6 to test versatility in handling conceptual ideas.

In some standard questions, similar to the routines in the instruction, the control students were able to perform as well or better, but overall the experimental students had a significant advantage (figure 2, table 2).



The sine of ∠A is a) 90°
b) 1.6
c) $\frac{1}{0.8}$
d) 1
e) 0.8
f) don't know

Figure 2: a standard question

Responses to Figure 2	Performance of Groups (%)			
	1	2	3	4
Experimental	84	88	62	59
Control	72	50	14	-

Table 2

In less routine examples the more able experimental students regularly performed better than the corresponding controls, with the less able experimental students more than holding their own (figure 3, table 3):

Draw a sketch of a triangle *ABC*, where ∠*C* =90°, *AB*=4 and sin*A*=cos*A*.
Use a ruler, and mark the values of angle *A* and angle *B*.

Note: this means the sine of angle *A* is the same as the cosine of angle *A*.

Figure 3 : A more versatile question

Responses to Figure 3	Performance of Groups (%)			
	1	2	3	4
Experimental	90	84	44	50
Control	55	44	45	-

Table 3

Table 4 shows the mean marks attained by each group on a school exam (at the end of the previous year), followed by the pre-test and post-tests 1 & 2, with the latter divided into standard questions (S) and more versatile questions (V). The experimental students perform at least as well on standard questions and outscore the control students on versatile questions. Those cases where the mean experimental score is greater than that of the control at a level $p<0.05$ are denoted by *sig*.

	Exam	Pre-test	Post-test 1			Post-test 2		
			S	V	Total	S	V	Total
E1	61	37	80	78	79	89	83	86
C1	70	39	80	47	64	86	50	68
			–	*sig*	*sig.*	–	*sig*	*sig*
E2	47	18	45	56	51	57	66	62
C2	48	20	46	38	42	39	41	40
			–	*sig*	*sig.*	*sig*	*sig*	*sig.*
E3	47	20	47	52	50	57	66	62
C3	49	21	31	28	30	24	23	23
			sig.	*sig.*	*sig.*	*sig.*	*sig.*	*sig.*
E4	39	*not given*	22	48	35	22	37	29
C4	39	*not given*	5	17	11	*not given*		
			sig.	*sig*	*sig.*			

Table 4: significance ($p<0.05$) of results of post-tests by group and type of question

Statistical differences between girls and boys

The performances of the various sub-groups are given in tables 5-8. In every group (except the least able, where full data was not gathered) from pre-test to (delayed) post-test, experimental boys improved more than control boys and experimental girls improved more than control girls. In each case the girls started out with a lower (or) equal score to the boys on the pre-test, but by the delayed post-test, the control girls improved *less* than the control boys, the experimental girls improved *more* than the experimental boys. In group 2 the experimental boys do not perform as well as the control boys on the first post-test. This was the one group where the experimental teacher found difficulty with the boys who both tried to compete in the use of the computer and also pressed the teacher to give them procedural ways of finding solutions. Even here, where the control pupils had extra teaching prior to the second post-test, the eventual superiority of the experimental pupils was shown. In particular, even though the control teacher tried the much-approved method of getting the pupils to reflect on and discuss their errors, he failed to bring about an improvement, indeed, the control pupils showed a marked deterioration in solving versatile questions involving the need to translate a word problem into a diagram.

Group 1	Experimental		Control	
	Boys (N=8)	Girls (N=17)	Boys (N=13)	Girls (N=16)
Pre-test	44 %	33 %	44 %	33 %
Post-test 1	75 %	82 %	70 %	61 %
Post-test 2	72 %	92 %	70 %	62 %

Table 5

Group 2	Experimental		Control	
	Boys (N=12)	Girls (N=13)	Boys (N=15)	Girls (N=11)
Pre-test	18 %	18 %	22 %	17 %
Post-test 1	43 %	57 %	50 %	31 %
Post-test 2	59 %	65 %	49 %	28 %

Table 6

Group 3	Experimental		Control	
	Boys (N=8)	Girls (N=13)	Boys (N =11)	Girls (N=10)
Pre-test	22 %	20 %	24 %	18 %
Post-test 1	56 %	46 %	24 %	36 %
Post-test 2	38 %	41 %	24 %	23 %

Table 7

Group 4	Experimental		Control	
	Boys (N=6)	Girls (N=6)	Boys (N=5)	Girls (N=9)
Pre-test	-	-	-	-
Post-test 1	34 %	34 %	15 %	8 %
Post-test 2	33 %	25 %	-	-

Table 8

Table 9 summarises the significant differences (*sig*) between experimental and control students using a two-way analysis of variance ($p<0.05$) to compare raw score in the two post-tests (in all cases the experimental students score more except for group 2 boys in post-test 1).

	Post test 1		Post-test 2	
	Boys	Girls	Boys	Girls
Exp v. Control 1	n.s.	sig	n.s.	sig
Exp v. Control 2	n.s	sig	n.s	sig
Exp v. Control 3	sig	n.s.	n.s.	sig
Exp v. Control 4	sig	sig	-	-

Table 9

This shows that the difference between experimental and control girls on the delayed post-test is statistically significant in all three groups tested but is not significant for any of the boys. In the immediate post-test only control group 3 fails to repeat the pattern. On the second post test the experimental boys outperform the girls only in group 4.

Conclusions

The experiment confirms the hypothesis that the experimental treatment using the generic organizer on the computer helped the experimental students to improve their performance compared with the control students. It also showed greater improvement in the experimental girls than the boys (except in the least able group).

The reversal of the hypothesis with the least able may be part of an overall trend: that the computer could help students (of either gender) lacking versatility in linking numerical to visual skills. This is consistent with the generally observed tendency for boys' performances to have a greater standard deviation, with more boys than girls performing well at the top end and more performing badly amongst the low achievers. It may be that those who gain an advantage from the computer compared with their peers include more able, less versatile girls and less able boys.

Whilst there was some evidence in experimental group 2 that the boys attempted to assume a dominant role, there is no evidence that this forcefulness translated into superior performances. The question as to whether the differences are social or genetic or due to other factors remain open. What is evident is a real improvement in the average and above average girls compared with the corresponding boys using the computer, which shows itself in an increased ability to think in a versatile way linking visual and numerical skills.

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LA DIMENSION DU TRAVAIL PSYCHIQUE DANS LA FORMATION CONTINUE DES ENSEIGNANT(E)S DE MATHEMATIQUES

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Résumé (1)

Depuis une dizaine d'années, j'ai développé des recherches à caractère clinique avec une orientation psychanalytique sur l'enseignement des mathématiques. A partir des hypothèses que ces recherches m'ont conduite à avancer, j'ai imaginé un dispositif pour la formation continue des enseignant(e)s de mathématiques, inspiré du groupe Balint, dans lequel ils (elles) peuvent effectuer un travail psychique en lien avec leur pratique professionnelle. Dans ma présentation orale, j'illustrerai le type de travail proposé dans ces groupes à partir de l'analyse d'une séquence enregistrée.

Etant Maître de Conférences en Mathématiques à l'Université de Paris-X Nanterre, j'ai bénéficié d'une expérience de l'enseignement des mathématiques à des étudiants en Psychologie et en Sciences de l'Education. La fréquentation d'étudiants le plus souvent en difficulté avec la discipline mathématique m'a amenée à réfléchir sur les questions, nombreuses et complexes, soulevées par l'enseignement des mathématiques.

Dans le cadre du statut d'enseignant-chercheur qui est le nôtre à l'Université, j'ai développé des recherches cliniques pour lesquelles je me réfère plus particulièrement à l'éclairage psychanalytique lorsqu'il s'agit d'interpréter les phénomènes qui se déploient dans l'espace didactique, tout en essayant d'intégrer dans ma compréhension de ces phénomènes l'apport de la théorisation effectuée en didactique des mathématiques.

Je n'ai jamais considéré ma recherche comme déconnectée de ma pratique d'enseignante de mathématiques, au contraire. J'ai toujours essayé de maintenir un dialogue fécond entre ces deux domaines : la recherche et l'enseignement, comme nous y invite notre statut d'enseignant-chercheur.

Lorsqu'on est enseignant(e), on sait beaucoup de choses sur la situation d'enseignement dans laquelle on habite jour après jour. Mais cette situation, on la connaît dpuis la place d'enseignant(e), c'est-à-dire de l'intérieur, dans l'urgence des décisions à prendre et sous la pression de l'action. Pour l'appréhender en tant que chercheur, il est nécessaire de prendre un certain recul. Pour effectuer ce "pas de côté", le chercheur est aidé par l'effort de théorisation qu'il accomplit dans le cadre de référence qu'il a "choisi" et par la mise en place de dispositifs méthodologiques permettant justement d'opérer cette distanciation.

Pour ma part, le choix qui s'est progressivement imposé à moi, en fonction de mon parcours épistémologique, a été de me référer à l'éclairage apporté par les théories psychanalytiques; ce qui m'amène à privilégier, dans la classe de mathématiques, le registre psychique des acteurs humains en présence ainsi que la dimension relationnelle de la dynamique nouée entre eux, autour de la transaction de l'objet de savoir.

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1 *My report will be presented in english and I will bring multiple copies of a longer version for distribution at the conference, in english too.*

Selon moi, dans la classe, l'enseignant est le leader de la situation: je veux signifier par là qu'il induit fortement ce qui va se passer au niveau de l'atmosphère de la classe. Bien entendu, les élèves interviennent aussi à ce niveau, mais c'est toujours en réaction aux propositions dont l'enseignant est totalement responsable.

Son moyen d'intervention privilégié reste la parole, même si des canaux de communication non verbaux sont aussi utilisés et même si le cours magistral est aujourd'hui souvent remplacé par la mise en situation d'activité des élèves.

Dans la situation didactique, au-delà de son rôle d'exposition d'un contenu de savoir mathématique, voici un sujet présent, le sujet-professeur, un sujet qui, assigné à la place de parole, va s'exposer à travers cette parole tout autant qu'il exposera. J'entends ici "sujet" au sens psychanalytique du terme, à savoir sujet de l'inconscient. Dans cette perspective, le sujet est un sujet divisé, l'inconscient étant à la fois l'acteur et le moteur de ce clivage psychique. De ce fait, au niveau de son discours didactique, le sujet-professeur est masqué à lui-même et cependant, "ça" parle de lui dans son discours sans qu'il le sache. Au niveau de ses décisions en situation didactique, il est soumis à des forces contraignantes internes qui le poussent à son insu, pour pouvoir survivre narcissiquement dans cet espace relationnel très risqué pour lui que constitue la classe; que l'enseignant soit entraîné au plan conscient par sa volonté épistémo-idéologico-didactique et que d'autre part, il soit soumis aussi à des contraintes externes à lui, telles que les contraintes propres au système didactique et les pressions institutionnelles, je n'en doute pas. Ainsi, l'enseignant est siège de tensions de toutes sortes. De toutes ces forces, va résulter un vecteur d'action dont l'orientation ne peut être qu'imprévisible à certains moments. Pour moi, ce vecteur résulte d'une organisation de compromis entre toutes ces exigences, organisation qui n'est, à chaque instant, que la moins mauvaise solution pour "faire face" comme on dit, pour "tenir", c'est-à-dire ne pas "s'écrouler" aux yeux des élèves et surtout à ses propres yeux.

Plus précisément, j'appelle contraintes intérieures, des contraintes que le sujet-professeur s'impose à lui-même, ou plutôt que sa part inconsciente lui impose, à son insu même. Nous entrons là dans le registre de ce qu'on appelle en psychanalyse la compulsion, plus précisément dans le registre de la compulsion de répétition, c'est-à-dire, d'un "processus incoercible et d'origine inconsciente par lequel le sujet (...) répète des expériences anciennes sans se souvenir du prototype et avec au contraire l'impression très vive qu'il s'agit de quelque chose qui est pleinement motivé dans l'actuel".

Pour moi, les contraintes intérieures pèsent lourd dans les choix de l'enseignant(e) au cours du déroulement de la classe.

Ces contraintes, bien que d'origine inconsciente, ne sont pourtant pas inaccessibles et c'est justement l'objet du travail psychique que de modifier les conditions internes qui donnent naissance aux tensions pour les dissoudre progressivement et obtenir un dégagement relatif de la force compulsive (2).

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2 D'après le Vocabulaire de la Psychanalyse de Laplanche et Pontalis, le travail psychique est un processus intérieur impliquant une activité du sujet (qui peut échouer) par lequel le sujet réussit progressivement à se dégager de l'emprise de la compulsion de répétition.

Je vais maintenant essayer de suggérer en quoi peut consister pour des enseignant(e)s un type de travail psychique apparenté à ce que je viens de décrire, en lien avec sa pratique professionnelle et quelles conditions sont nécessaires pour qu'il puisse s'effectuer, sans pour autant se situer dans un processus de cure psychanalytique.

Le dispositif que j'ai imaginé est conçu pour la formation permanente des enseignant(e)s de mathématiques dans la perspective d'un développement personnel. Il s'agit d'un groupe qui réunit des enseignants et des enseignantes de mathématiques de lycées et collèges. Le fonctionnement de ce groupe est inspiré par la technique du groupe Balint. Je le pratique aujourd'hui depuis neuf ans avec des effectifs restreints, allant de quatre à sept personnes selon les années. Ce dispositif consiste en un groupe dans lequel les échanges verbaux sont centrés sur les implications psychologiques de la pratique d'enseignant(e) de mathématiques. A partir des récits effectués par les participants d'incidents survenus dans leur classe ou encore à partir des questions qu'ils se posent au sujet de leur pratique, une exploration se tisse au fur et à mesure des rencontres. Les séances ont lieu tous les quinze jours selon un horaire régulier. Elles durent deux heures. Chaque enseignant participant se voit progressivement en mesure de repérer ses propres modèles de comportement - ses patterns de comportement, dirait Balint - les plus fréquents en situation didactique ; non pas les comportements qu'il pense avoir ou qu'il a l'intention d'avoir selon ses convictions idéologiques et pédagogiques mais les comportements qu'il a effectivement dans la classe et que le travail de groupe a des chances de lui révéler. A condition que lui-même soit prêt à effectuer un travail de ce

type, autrement dit qu'il ait la possibilité de se laisser aller au minimum à ses propres associations. Il est à noter que le fait qu'il soit volontaire pour entreprendre ce questionnement est une condition nécessaire mais non suffisante. Car, entre la volonté de l'enseignant, sa décision d'"aller y voir" et un réel travail de prise de conscience, va s'interposer son inconscient, avec tout le poids du refoulement et des résistances qui vont imposer leurs censures et aveuglements divers. Cependant, avec le temps, beaucoup de temps, et une conduite du travail adéquate, quelques insights(3) surgissent comme un voile qui se déchire l'espace d'un instant de lucidité pour que chaque participant(e) puisse affronter sa véritable image d'enseignant(e), dont les contours vont progressivement lui apparaître, par contraste avec les autres enseignants du groupe. Pour cela, il va de soi que l'atmosphère dans le groupe se doit d'être propice, facilitante, suffisamment chaleureuse et contenante mais en même temps mobilisatrice et non complaisante. Au-delà de leurs patterns de comportements, les participants du groupe essaient d'identifier, avec mon aide, les craintes, les angoisses à l'oeuvre ou les gratifications et bénéfices recherchés qui déterminent ces comportements. Ce qui les amène immanquablement à réfléchir et à s'interroger sur leur propre rapport au savoir mathématique issu de leur histoire personnelle et de leur formation, et de l'évolution de ce rapport depuis qu'ils ont commencé à enseigner. L'objectif étant toujours ramené à l'acte didactique lui-même : comment leur propre rapport au savoir mathématique s'infiltre dans la situation didactique,

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3 Pour Hanna Segal, dans "Délire et créativité", l'insight consiste en "l'acquisition d'une connaissance au sujet de son propre inconscient, connaissance acquise au moyen d'une expérience consciente".

quelles attentes sont les leurs de ce fait, que projettent-ils sur leurs élèves ?

Au fond, chaque enseignant(e) se fabrique au fil des ans des modèles de comportement qui réalisent pour lui un compromis acceptable : une sorte de confort, de ceinture de sécurité. Toucher à cet équilibre de compromis - et toute modification de type didactique, la plus simple soit-elle, y touche - est coûteux pour l'enseignant. Cela l'expose à des passages inconfortables même si, à moyen terme, il sait pouvoir y réaliser une économie. Une économie d'énergie par exemple. En effet, lorsque ce pseudo-équilibre est maintenu à grands renforts de mécanismes de défense ou au prix d'un clivage rigide entre le soi privé et le soi professionnel ou entre l'image idéale de soi et l'image du soi renvoyée par les élèves, beaucoup d'énergie psychique est dépensée à maintenir cet état métastable. Dans ce cas, alléger le système défensif, atténuer les clivages libèrent une énergie qui est remise en circulation pour faire autre chose, par exemple réussir des changements didactiques. Cette fluidité retrouvée, un sentiment de liberté est rendu au sujet malgré le poids du déterminisme psychique et on peut alors espérer que la compulsion de répétition, si prégnante dans la structure d'enseignement, sera un peu moins à l'oeuvre. Et du même coup, l'enseignant a peut-être moins de peine à reconnaître les limites de son action et à assumer sa non toute-puissance.

Dans ma présentation orale, j'illustrerai ce que je viens d'avancer à l'aide d'exemples puisés dans une séance de groupe enregistrée en juin 1987, après une année de travail régulier avec quatre professeurs de mathématiques, deux femmes et deux hommes. Le matériel que j'apporterai sera centré sur l'apport de l'un des

participants à cette séance; il me permettra d'illustrer le thème du rapport à la parole et aussi de suggérer comment les scénarios fantasmatiques peuvent déterminer le comportement de l'enseignant en situation à son insu en fonction du lien transférentiel qu'il a noué avec les élèves.

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DRAWING A DIAGRAM: OBSERVING A PARTIALLY-DEVELOPED HEURISTIC PROCESS IN COLLEGE STUDENTS

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Through structured individual interviews, we are able to observe and describe in considerable detail the problem-solving capabilities of college students with regard to the heuristic process, "drawing a diagram."

Introduction and General Description of the Study

"Drawing a diagram" has long been recognized as a valuable heuristic process for problem-solving in mathematics (Polya, 1957, pp. 103 - 108), and has recently received some research attention (Schoenfeld, 1985, p. 195; Gordon, 1983; Gonzales, 1986; Simon, 1986; Bodner, 1990). Our interest here is the complex interplay of processes involved in students' use of diagram-drawing during problem-solving. This paper describes the problem-solving processes used spontaneously by college students when solving a story problem during individual interviews, as well as their responses to minimal structured suggestions or "hints." The data were obtained as part of the first author's dissertation study of 40 incoming first-year students enrolled in developmental (non-college credit) elementary algebra classes at Rutgers University. Twenty-two of these students participated in individual "draw a diagram" (DAD) interviews, while 18 participated in a different interview for comparison purposes. The interviews, which were videotaped, were conducted prior to class instruction in solving mathematical word problems.

The script used by the clinician during the diagram-drawing interviews was modeled structurally on a script described by Goldin (1985) for a different heuristic process, "think of a simpler problem." It consists of six sections: [I] Introduction; [II] Understanding the terminology and concepts: rectangle, width, length, dimensions, and perimeter; [III] Presentation of the problem:

> "The length of a rectangle is one inch greater than twice its width. The perimeter is 26 inches. What are the dimensions of the rectangle?"

Here *the clinician encourages free problem-solving as much as possible* and does not interrupt the subjects; [IV] Guided use of the heuristic process, "Drawing a Diagram." At this point the clinician gives minimal heuristic suggestions, *but only when students are unable to continue*: [V] Presentation of diagrams The intent here is to present a diagram only as a last resort; in fact, this was not necessary in any of the 22 interviews; and [VI] Looking back The clinician asks appropriate non-leading questions to learn more about the student's thinking, or to explore why an expression may have been written Free problem-solving is encouraged as much as possible throughout the interview

The rationale for such a structured script is the following first, we are able to learn the maximum about the student's own spontaneous problem-solving strategies and heuristics Second, when an obstacle occurs we provide just that minimal heuristic suggestion needed to overcome the difficulty, allowing us to learn more about the student's capabilities and problem-solving cognitions through *subsequent* spontaneous behavior In this fashion, we can explore and describe in great detail the diagram-drawing capabilities used spontaneously, the capabilities manifested only when they are suggested through "hints", and those which do not occur at all The idea is to map out a complex structure of heuristic subprocesses, rather than treating "drawing a diagram" as a single "all-or-nothing" capability

Overview of Highly- and Partially-Developed Diagram-Drawing Capabilities

The following advance criteria were used to categorize the processes spontaneously evidenced

Whether or not a rectangular diagram was drawn spontaneously during the solution attempt;

Whether or not the diagram was spontaneously labeled during the solution attempt, and on how many sides or dimensions of the rectangle;

Whether or not a correct solution was obtained during the initial spontaneous, uninterrupted period of problem solving.

Of the 22 DAD subjects, 15 spontaneously drew diagrams (rectangles) for the problem Ten of these 15 also labeled their diagrams spontaneously in some way One of these 10 students Mandy, labeled the four sides of her rectangle with *numbers* obtained by guessing or trial and error The remaining nine of the 10 students drew rectangles and labeled at least one side with a variable quantity expressed in terms of letters

Four of these nine labeled their rectangles spontaneously on all four sides with variables two of the four students (Abby and Leroy), used valid variable assignments and the other two of the four (Paula and Sarah) used an incorrect variable assignment Abby and Leroy obtained correct solutions to the problem Paula and Sarah both had incorrectly labeled the rectangle's width as 2N or 2X, instead of N or X Paula did not solve the problem correctly during this spontaneous phase She set up a "correct" equation using her incorrect variable expression for the width, but then encountered an obstacle in combining like terms (using formal notation): when adding 2X plus 2X she obtained 4X-squared. Sarah, who changed her method to trial and error after labeling the dimensions, solved the problem by guessing the answers.

Four other students out of the nine (Alice, Naomi, Roger, and Shirley), had spontaneously labeled their rectangles on only two sides. All four of them attempted to set the sum of the variable quantities labeled on just the two sides equal to the value of the perimeter. The ninth student, Rodney, spontaneously labeled only one side of the rectangle and he set this quantity equal to the perimeter value. Our observations are consistent with the conjecture that these students made use of the appearance of the diagrams they constructed in setting up their (correct or incorrect) equations, in that all the students demonstrated understanding of "perimeter" earlier in the interview.

Of the five students who drew unlabeled rectangles (out of the 15 who spontaneously drew rectangles), one (Dick) calculated a semi-perimeter and two (Bill and Mary) set the variable quantity describing one dimension equal to the perimeter. The other two (Nancy and Vicky) did not write any equations initially.

The remaining six students who did not spontaneously draw diagrams made errors rather similar to those made by the students who drew partially-labeled diagrams. Three students (Jane, Jason, and Millie) attempted to equate the perimeter with the sum of two dimensions and two students (Dana and Toby) set the perimeter equal to the variable quantity derived for one of the dimensions. The sixth student (David) used trial and error to guess answers.

In all, four of the 22 students in the DAD group were successful in their initial, spontaneous solution attempts: Abby and Leroy correctly and completely labeled rectangles; Sarah, drew a completely labeled rectangle using an incorrect variable quantity for the width, but then changed her method to trial and error, successfully guessing the solution; and Denise drew no diagram for the main problem, although she had drawn two rectangles during Sec. II, earlier in the protocol. Of the 18 students who were initially unsuccessful: Paula set the sum of the four sides equal to the perimeter, however she was unable to combine like terms; eight calculated a semi-perimeter; five equated the variable quantity of one dimension with the perimeter; and four were unable to write any useful equations. An algebraic approach was the predominant method, used spontaneously by 19 of the 22 students, while trial and error was used by the remaining three students.

The Problem-Solving Processes of Two Students

Next we describe in more detail the problem-solving processes of two of the students

(Roger and Mary). The descriptions come both from the free problem-solving phase of the interview, prior to any hints or suggestions given by the clinicians, and from problem-solving in response to minimal suggestions. Excerpts from the protocols are provided below, following the discussion.

Early in the interviews, before being asked to solve the main problem, both students demonstrated an understanding of the terminology (Sec. II). Throughout her interview, Mary consistently labeled diagrams only with her answers, using them mainly to *check* her answers (See Protocol A and the accompanying schematic: quotes #1, 4, 5, 7). She drew six rectangles in all and labeled each with numbers. Only later, after being asked, did she draw a rectangle labeled with variables, L and W. She used the > sign in place of the + sign four times, whenever translating the phrase "greater than" from the problem statement. She realized this was "wrong" but continued to write it anyway (quote #2). When writing the perimeter equation, she realized that she had omitted the width term but then became confused, thinking that if she added "2W" to "2(1+2W)" there would be "too many W's" in the equation (quotes #3, 6, 8). After many questions, she solved the problem algebraically and labeled her last diagram with her answers.

In contrast, Roger drew a total of three rectangles (one for each solution attempt, see Protocol B and the accompanying schematic) and appeared to rely on these diagrams for setting up his equations (quotes #9, 10, 11), calculating the values of the dimensions (quotes #10a, 11a), and checking his solutions (quotes #10b, 11b). During his first attempt, he labeled only two sides of the rectangle (one length and one width) and subsequently set the sum of only those two sides equal to the perimeter. He recognized this as an error during his second attempt, after he had labeled all four sides of the rectangle (quote #12); however, he still labeled the width "2W." Only during the "Looking Back" stage of the script (Sec. VI), did he validly label all four sides and arrive at a correct solution.

The broad "diagram drawing" strategies of these two students differed in their planning reasons for using the diagram. Mary had no trouble drawing rectangles but she appeared to use her diagrams only as an aid for checking the solutions she obtained by trial and error. Her initial algebra did not work, and she spontaneously decided to rely on trial and error. Only after considerable later dialogue did she successfully use algebra, but then without referring to her diagram. She was able to label a rectangle with variables and refer to it when prompted, but she referred spontaneously to the rectangles she drew only to verify the correctness of her number values. When she did label her rectangles, she labeled all four sides. Roger on the other hand used his diagrams throughout the entire solution attempt for setting up his equations, calculating the values of his dimensions, and checking his solutions. In his first attempt, since he had labeled only two sides of his rectangle, his reliance on the labeled diagram may have actually "led him astray". Using his diagrams however was not sufficient to overcome the persistent error in his having labeled the width as "2W". This error occurred consistently until the "Looking back" stage of the script. Roger's diagram-drawing instruction in high school had made enough of an impression for him to remark several times in the course of his interview that one "should always draw a diagram."

From these protocols it is possible for us to construct *a detailed sequence of competencies,* based on each student's observed behavior, and to characterize his or her capabilities in terms of whether the student *spontaneously* uses the process, whether the student uses the process *when prompted*, and whether the student's use of the process (spontaneous or prompted) is *successful* (in the sense that the process results in a valid, usable outcome). For instance, in the case of Mary and Roger, the following is a *partial list* of processes and/or sub-processes that were observed. The complete protocols and competency charts for all 22 subjects may be found in Bodner (1990).

Attempts a solution
Decides to draw a diagram
Draws an arbitrary rectangle
Identifies the length and/or the width on the diagram
Makes use of the drawn rectangle
Assigns variable(s) to represent the width (e.g., W) and/or the length (e.g., L, 2W+1)
Uses a related variable for one or both dimensions
Checks the variable assignments for reasonableness
Labels one or more of the dimensions on the diagram (with variables, or with numbers)
Attempts to write an equation involving the dimensions and the perimeter
Attempts to write an equation to describe the length
Decides to use trial and error
Solves the problem by algebraically manipulating equations
Checks answers for reasonableness
Verifies that the sum of the four dimensions equals the perimeter value
Concludes that an answer is correct
Concludes that an answer is incorrect
Explains the solution method coherently
Provides coherent reasons for using a diagram in the solution attempt
Corrects own conceptual misunderstanding

Instruction on how to draw a diagram does not alone guarantee students' success with verbal mathematics problems. But by specifying the sub-competencies brought to bear in "drawing a diagram", we are better able to understand the knowledge used by students, including their executive-level heuristic strategies while working on the problem (Goldin, 1984). Having these insights into the way thinking and learning take place allow us to create a "prescriptive version" (Schoenfeld, 1987, p. 18) of the "drawing a diagram" heuristic process, which in turn may help students become more proficient at diagram drawing and problem solving in general.

Protocol A: Excerpts from Mary's Protocol

1 "Well if the numbers were right and they came to 26, (M writes 12.5 and 11.5 on the four sides) I would put the length where the length is supposed to be and the width where the width is supposed to be. That was if you were going to ask me for the perimeter. But you just wanted the dimensions, so ..." (M has drawn two more rectangles. The first she labeled with various numerical values. The second, prompted, she labeled L and W.)

2 (M has written 26 = 1>2L) "That is wrong. I don't know. I guess I should have a plus because if they're saying greater, I'm just thinking greater than, less than, so ... I can put it in like that."

3 "Okay I figured out that L equals one plus two W, so since I have two of them it would be two times one plus two W for both. And they would come to 26, but then we're forgetting the width, not unless we're gonna' pick it up here. I'm not quite sure. ... We already have it in the equation, then so I guess we don't have to get it again. We already have it once."

4 "If the width would be six ... draw another figure, it would be six, six ... (M labeled her rectangle with 3 and 6 on four sides) ... so then I guess that would have to be three but that's not enough, only coming to 18."

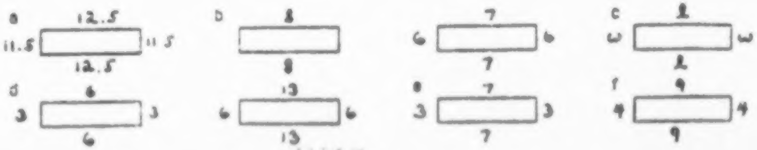
5 "Umm. Oh, it's in the wrong spot. Okay, okay. It'll go there (M labeled the two widths with 6). That's the width. ... so that means the length is gonna' be 13 (M labeled the two lengths with 13). 13 and 13 that's 26 right there. So that wouldn't work if I had 13, because that would be too much. Not unless they mean, three and three just divide this by two to get the width of each one which would be three, so it would be seven? (M labeled all sides with 3 and 7) That's not enough either. That's only 20."

6 "I just thought that I would substitute L which I got here, but I forgot to add on. I don't know if I already need it. I have it in my equation so I guess I should continue it out. Two W, but I'll have too many. 26 equals two L plus two W and then 26 equals two plus one plus two W. I'll just add on the two W but I don't know. Oh, all right, 26 equals two plus four W plus two W. See, I'll have too many, oh wait a minute, never mind. 26 equals two plus six W minus two is 24 equals six W. I have a new one now."

7 "I'll draw a diagram again (M labeled the four sides with 4 and 9). The width would be four and it'd be nine? That's right, for some reason." (M did her algebraic calculations before drawing her sixth rectangle.)

8 "I don't know why I left it out to begin with, because if I have the equation I should have used everything. To begin with I just left that whole step out completely. Because, I don't know, it was because I thought that there were too many, it's stupid but, too many W's. Because there was, there was two of them, but I forgot, it came back that you could add them together again."

Mary's Diagrams (schematic)



Protocol B: Excerpts from Roger's Protocol

9 "The length of the rectangle is, okay (R drew a rectangle). So this is length, that's width (R pointed to the rectangle and wrote L and W in lowercase letters on two of the sides of the rectangle). Okay, the length is greater than twice the second (R labeled both the width on the right hand side and the uppermost length with n) The length is one inch greater than twice the width (R wrote a 2 in front of the n already labeled on the width). Ooh! One inch greater than twice the width (R wrote in a 2 in front of the n on the length side and added on a + 1 after the n). That'll be one inch greater than twice the width, so if that's twice the width, that's two, I'll put two W. Now I'm getting confused (R changed the two n's to W's on the two sides already labeled). So it's like that man (R started to write an equation with 2W + 1). I'll say, so it's telling me the dimensions already, so I would say two W plus one by two W, that's what I would say. But not only is askin' me the dimensions of it, they ain't askin' me to solve the problem, but I will solve the problem too, just in case (R continued the equation by writing + 2W). I'm not sure if it's right or not but I'll see anyway. So I'm sayin' that's four W plus one equals 26. Bring that over. Oh no. It ain't working it out. That's one, that's 25. Four W. Oh that's four that five, six, six that's not working. No. So I will go with this answer. So I would say two W plus one by two W." (R writes "by" on his rectangle, see Roger's Diagram a).

10 "Okay, so, if I'm adding this, so, I'm seeing, well, like for in place this (R wrote 6-1/4
10a above the W on the rectangle and then wrote 13-1/4 as a label for one of the lengths).
Okay, I guess I'm not sure. (R wrote 6-1/4 above the W on the rectangle and then wrote
10b 12-1/4 as a label for one of the widths.) Okay, so you said it's 12 by 1/4 and 13 by 1/4.
Now if you add that, I'll know. I'm wrong. I see it already."

11 "I'll read the problem again, draw a rectangle. It's one inch greater than twice its width.
I have to repeat that in my head. I could have been wrong when I said two W plus one.
I could have said two W (R labels one width with 2W) and W plus one (R labels one
length with W + 1). But then again, it says twice its width. So I figure you put 2W plus
one. (R inserted a 2 as the coefficient on the W on the length). Let's see okay. That's
right! I, I believe that is right. Now, I do not know why this do not come out like I
want it to be. (R labeled the remaining two sides with 2W and 2W plus 1). But hold
up! If I add all this, if I add that and that (R pointed to all four sides). I'm not sure,
11a I'm wrong. I see it already plus four W plus two. I see I'm wrong already, plus two let's
see. 4W plus two equals 26. I'm wrong. Oh wait a minute. I'm wrong. I'm wrong.
11b That's three. No it's not working, seven (R labeled the rectangle with a 3 above the W
and then labels with 7 on two sides) That's this three and that's six (R labeled one
width.) plus three that's six (R labeled the other width) and seven (R labeled the last
length), 14, and 12? Ahh, it's not working. Oh, wait, hold up! I rest my case." (see
Roger's diagram b)

12 "Can I explain this and how I messed up before? Okay. Now this one I made a mistake in my multiplication, well not really the multiplication, my perimeter adding. Because when I did my perimeter adding I did not put the two W on each side. What I did was I just added this, the length and the width but I did not add the other side, the other length on the other side. I just let that be, 'cause I was figuring that the length times the width is that's it. So I just let that be, so but in the second problem I did, I added all the sides together, okay? It came out to W. Four W plus four W plus two equals 26. That's what I came up with. Now when you get to three you put the W, put the three in place of the W, when you do that you add up like this, see? That's two W plus one and this one is two W. That came out to six on each side. So you add up all the sides, as in doing the perimeter, and you come up with 26." (R draws the rectangle during Section VI of the script, see Roger's diagram c.)

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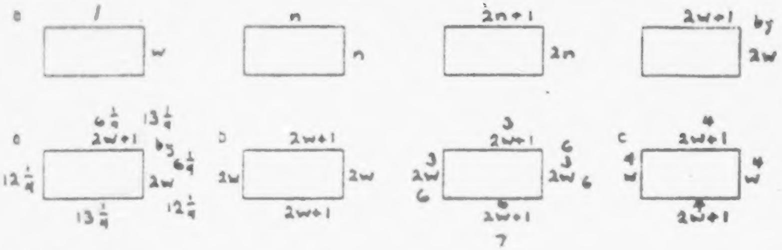
Roger's Diagrams (schematic)



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THE ACTIVE COMPARISON OF STRATEGIES IN PROBLEM-SOLVING: AN EXPLORATORY STUDY

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Summary

The aim of the research we are reporting was to understand to what extent the comparison of strategies affects children's attitudes and problem-solving skills. As regards this problem, we have observed that in arithmetical problems the comparison of strategies induce almost all the pupils to construct (or choose) the same strategy and not to search for alternative ones. On the other hand, in geometrical problems it seems to stimulate the search for different strategies, and the progressive development of some geometrical concepts. In this report we study its effects on the strategies used by two groups of fifth graders in order to solve some sequences of arithmetical and geometrical problems and make some conjectures to explain the differences we have observed.

1. INTRODUCTION

In the Genova Group's Project the active comparison of pupils' problem-solving strategies (as sketched in 2.2.) has proved very important related to the following purposes:

- to promote the construction and the diffusion of more effective strategies [Boero et al., 1989];
- to encourage children to make and check hypotheses; in particular, poor problem-solvers can explicitly produce hypothetical reasoning in this context before than in others [Ferrari 1989].

As for the effects of the active comparison of strategies, we have observed in a large number of classes some important differences between arithmetical and geometrical settings. In arithmetic it seems to foster the diffusion of only one strategy for each problem. In geometry, on the contrary, it seems to encourage the planning of alternative strategies for each problem. In this report we study the effects of the comparison of strategies more carefully in order to explain the reasons for this different behaviors. This may give some interesting piece of information on the ways pupils build or choose their strategies.

2. EXPERIMENTAL DESIGN

2.1. The context

The 2 classes we have taken into account were working with the Genova Group's Project. Related to the object of this study, we point out the following characteristics of the Project (a further information on the educational context in which the research is included can be found in [Boero,1988], [Boero,1989], [Boero - Ferrari - Ferrero, 1989], [Ferrari, 1989]):

- long-term planning of educational work in all the subject areas (usually with the same teacher);
- familiarity with verbalization processes, due to activities like reporting in written form the strategies used to solve a problem as well as a discussion or an experiment performed in the classroom;
- the stress on the construction of linguistic competence to describe procedures or relationships among facts; in particular, the stress on the mastery of connectives (before, after, when, while, because, if…then, …);
- the acquisition of the algorithms for addition, multiplication, subtraction and division at the end of processes of progressive schematization and generalization of the euristic strategies proposed by the pupils.

Furthermore, the search for alternative strategies is strongly promoted: for example, each pupil, after reaching a solution, is requested to find another strategy on a different sheet.

2.2. The active comparison of strategies

In our classes it is usual, after solving a problem, to report, discuss and compare pupils' strategies. By 'active comparison of strategies' we mean a more structured work which is carried out after some of the problems, organized as follows. The teacher gives the pupils a sheet where are reported some of the strategies used by them for that problem. The strategies are not reported in the same form they were written down by the pupils, and generally they are reported incompletely (e.g. without any calculation or without any explanation). Each pupil is asked to reconstruct the strategies in written form, to recognize whether one of them is the same he has performed and to compare them, pointing out the differences.

2.3. The sample

We have gathered a large amount of protocols from 2 classes of grade 5 (41 pupils) where the active comparison of strategies have been widely employed. These classes have experienced the Genova Group' Project since first grade. The materials we have analyzed concern the resolution of a lot of arithmetical and geometrical problems. The pupils were requested to report, in written form, the procedure they have performed. On account of the general characteristics of our Project (see 2.1.) this request is usual for our pupils and then it has not caused any specific problem. The teachers usually support the pupils in trouble, for example explaining them the data.

The materials have been analyzed with particular regard to:

- the correctness of the procedures and the explanations reported
- the strategies used by the pupils
- the quality of the reports, from the point of view of language, with particular regard to improvements in the use of words (which may be a sign of some new conceptual attainment); this quality does not necessarily coincide with general linguistic competence, since there are pupils who master language quite well in many situations but do not display their mastery when reporting their procedure.

2.4. The arithmetical problems

We have analyzed the protocols related to 20 arithmetical problems (mainly, problems related to money, as the comparison of the cost of a journey using train, car or plane, or the determination of the cost of some goods produced in class, or the daily profits of a factory during the Industrial Revolution). The problems have been administered from november 1989 to march 1990, and their 'structural' hardness (as for numbers and operations involved) was approximately the same. They were related to different problem situations with the exception of a sequence of 5 similar problems (problems A1-A5) such as the following:

PROBLEM A1: *"Mr.Smith has a factory with 20 looms. Each loom produces 2 rolls of cloth daily. The cost for raw materials is 2 guineas per roll, and the selling price of each roll is 3.2 guineas. In the factory are employed 20 workers, and the daily pay of a worker is 0.3 guineas. What are the daily profits of mr.Smith?"*

Table 1: the data in problems A1-A5

Data \ Problems	A1	A2	A3	A4	A5
n. of looms	20	20	20	120	1
n. of workers	20	20	10	120	1
units of product (per loom, daily)	2	1	14	1	120
selling price (guineas, per unit)	3.2	4	2.5	2	1.5
raw materials cost (guineas, per unit)	2	2	1.5	0.8	0.8
wages (guineas, per worker, daily)	0.3	0.3	0.1	0.2	0.2

After each problem the pupils' strategies have been informally reported and discussed. After problem A3 a session of active comparison of strategies has been proposed. During the sequence A1-A5 no other arithmetical problem has been administered.

2.5. The geometrical problems

We have analyzed the protocols related to 15 geometrical problems, administered from november 1989 to march 1990. The problems concern mainly the area of real surfaces (such as the classroom) or figures drawn on a sheet. Pupils were requested not only to find the area of some figure, but sometimes also to construct a figure of a given area, or a figure which is twice (or half) the area of another, and so on. After some problems of this kind we have planned a sequence of 5 problems; the task was to find out the area of the following figures:

G1 a right-angled triangle; G2 a isosceles triangle; G3 a parallelogram;

G4 a right-angled trapezium; G5 a double right-angled trapezium.

When solving these problems, pupils generally did not know the formulas to find out the area of regular figures. Few pupils were aware of the formulas for the area of a rectangle, and/or of a triangle and/or of a parallelogram. This is the reason why some of them, for example, use their knowledge about the parallelogram to find out the area of triangles. After each problem the strategies used by the pupils have been informally reported and discussed. Active comparison of strategies has been planned after problems G2 and G4. During the sequence no other geometrical problem has been administered.

3. ANALYSIS OF THE DATA

3.1. The strategies in arithmetical problems

To solve the problems of the sequence A1-A5 (that is, to find the daily profits of a factory) the pupils have used one of the 3 strategies sketched below:

= strategy A: total takings - total expenses = total profits;
= strategy B: profits per machine × number of machines = total profits;
= strategy C: profits per unit of product × number of units = total profits.

3.2. Some results on the arithmetical problems

In all the arithmetical problems pupils seemed not very glad to search for alternative strategies; even clever problem-solvers, requested to build another strategy, generally did not succeed, or proposed another time the strategy they had already found. As to the sequence A1-A5 only 7 pupils (all good problem solvers) have produced at least one time 2 strategies (see table 2) and none more than 2.

Table 2: number of pupils who have produced more than one strategy

Problem A1	Problem A2	Problem A3	Problem A4	Problem A5
7	7	4	0	0

Strategy A has been clearly preferred through all the sequence, whereas the other strategies have been progressively neglected (see table 3). It is worth to note that after the active comparison of strategy all the pupils have used strategy A and no pupil has changed strategy once used strategy A the first time.

Table 3: number of pupils who have used each strategy in their first answer

	Problem A1	Problem A2	Problem A3	Problem A4	Problem A5
Strategy A	28	34	37	41	41
Strategy B	9	5	4	0	0
Strategy C	4	2	0	0	0

Correctness and autonomy have improved during the sequence, as shown in table 4. This result has probably been affected by the similarity of the 5 problems.

Table 4: number of pupils who give at least one correct answer without any help

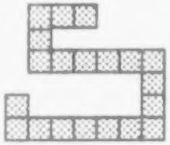
Problem A1	Problem A2	Problem A3	Problem A4	Problem A5
20	26	29	27	31

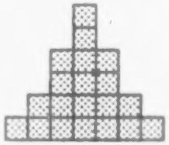
The results related to the other 15 (different) problems confirm the findings of table 2: the average number of strategies built by pupils is very low (about 1.1 strategy per problem) and only clever problem-solvers can sometimes build more than one strategy. In general, the particular problem situations seem not to affect very much the number of strategies used. We have not found significant improvements as to autonomy in the design of resolution procedures (although the difficulty of the problems was almost the same); in fact the number of pupils giving a correct answer without any help is increased less than in problems A1-A5, probably because of the variety of the problem situations involved: we have found about 20 correct answers to the first problems and about 25 to the last ones.

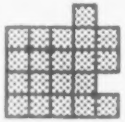
It come out some correlation between the correctness of the answers and the quality (on the linguistic ground) of the verbalizations: the reports of all the children who make serious mistakes or need substantial help by the teacher are very poor (not necessarily incorrect) from the syntactical point of view, with few connectives (at most, some temporal connective). This sort of correlation has not been found in geometrical problems.

3.3. The strategies in geometrical problems

To build a figure of area 20 squares (not a rectangle), the pupils have used a large number of strategies. We report only three of them, for example.

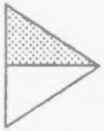


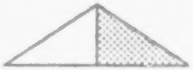


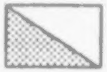


The first and the second figure have been drawn iteratively and are based on the counting of the squares. As to the third, the pupils have drawn a rectangle of 20 squares and then have transformed its shape, to fulfil the second condition of the task, without affecting its area.

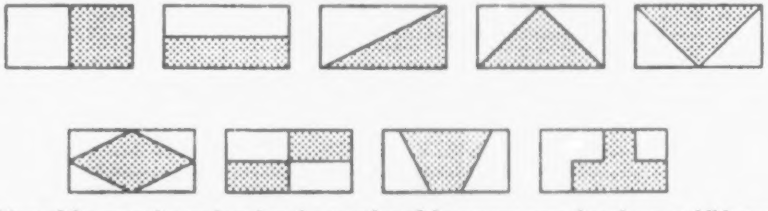
To build a figure of area twice the area of a right-angled triangle the pupils, on the whole, have used the three strategies sketched below.





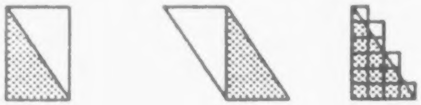


To build a figure of area half the area of a rectangle the pupils have used the nine strategies sketched below.



Most of the strategies are based on the counting of the squares even though some children was aware of the formulas to find the area of a rectangle or a triangle.

Now we report the strategies used by the children to solve the subsequent problems of the sequence G1-G5. As to problem G1, they have used the 3 strategies sketched below.

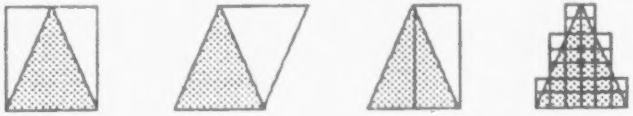


strategy 1a — $A_R:2$

strategy 1b — $A_P:2$

strategy 1c — counting squares

Problem G2 has been solved using 4 different strategies.



strategy 2a — $A_R:2$

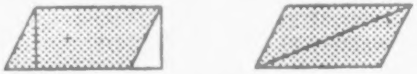
strategy 2b — $A_P:2$

strategy 2c — A_R

strategy 2d — counting squares

Notice that some children have already known how to find the area of a parallelogram (strategies 1b, 2b), whereas others are still counting the squares (strategies 1c, 2d). Strategies 2a, 2b, and 2c have been used for the 'active comparison' planned at this point of the sequence.

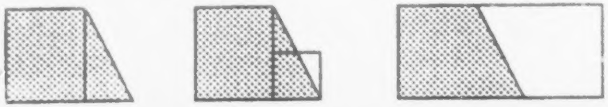
Problem G3 has been solved using only two strategies.



strategy 3a — A_R

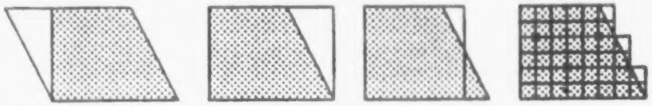
strategy 3b — $A_T \times 2$

To solve problem G4 the pupils have used 7 different strategies.



strategy 4a *strategy 4b* *strategy 4c*

$A_R + A_T$ $A_{R1} + A_{R2}$ $A_R : 2$



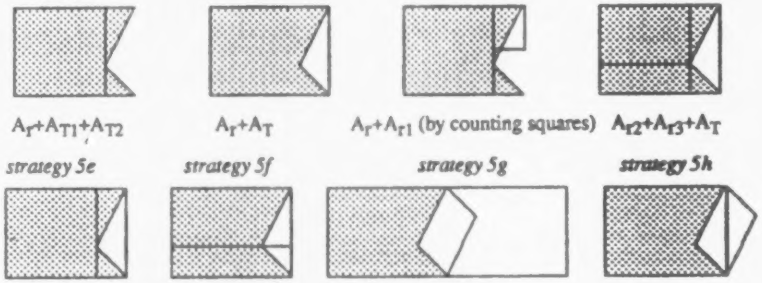
strategy 4d — $A_P\text{-}A_T$

strategy 4e — $A_R\text{-}A_T$

strategy 4f — A_R

strategy 4g — counting squares

At last, to solve problem G5 they have used 8 strategies.



strategy 5a — $A_r+A_{T1}+A_{T2}$

strategy 5b — A_r+A_T

strategy 5c — A_r+A_{r1} (by counting squares)

strategy 5d — $A_{r2}+A_{r3}+A_T$

strategy 5e — $A_R\text{-}A_T$

strategy 5f — $(A_{R1}\text{-}A_{T1})+(A_{R2}\text{-}A_{T2})$

strategy 5g — $(A_{DR}\text{-}A_P):2$

strategy 5h — $A_R + A_P:2$

3.4. Some results on geometrical problems

Children were generally glad to search for other strategies. The answers seem to be affected by the particular shape of the figure involved. For example, in problem G3 almost all pupils have used strategy 3a and very few have built more than one strategy, whereas in the other problems of the sequence G1-G5, as well as in most of the geometrical problems many pupils, and all the clever problem-solvers, have built more than one strategy. This may depend on the previous knowledge of some pupils about parallelograms or even on the simmetry of the figure. In the sequence G1-G5 the number of pupils who have built at least one correct strategy without any help has increased, in spite of the rising complexity of figures, as well as the number of those who have built more than one correct strategy (table 5). The improvement in building correct strategies is a consequence of two different processes: poor problem-solvers generally learn to use strategies performed by others in some previous problem (or reported in the comparison of strategies), whereas clever problem-solvers become able to project new ones. In problem G5 21 pupils have produced at least one strategy different from those used for problem G4; 11 pupils (and in particular 5 very poor problem-solvers) can use strategies similar to those reported in the comparison of strategies, which they cannot perform without help in the previous problems.

Table 5: number of pupils who have built at least one (a) or more than one (b) correct strategy without any help

	Problem G1	Problem G2	Problem G3	Problem G4	Problem G5
a	24	27	31	28	37
b	11	14	3	18	24

The average number of strategies used for each problem has increased, with the exception of problem G3. In particular, in the last problem of the sequence (G5), after the active comparison related to problem G4, 8 different strategies have been used (table 6).

Table 6: number of pupils who have used each strategy in problem 5 in some answer (a) or in their first answer (b)

	5a	5b	5c	5d	5e	5f	5g	5h
a	17	15	2	1	16	5	10	3
b	12	9	2	0	9	4	3	2

The strategies based on the counting of squares are progressively neglected (table 7).

Table 7: number of pupils who use counting of squares in their strategies

Problem G1	Problem G2	Problem G3	Problem G4	Problem G5
21	20	4	9	3

The correctness of the answers and the quality (on the linguistic ground) of the verbalizations seem not much correlated: we have found a lot of correct strategies very poorly reported and, on the other hand, very well-written reports of uncorrect ones.

The results related to other 10 geometrical problems generally agree with the results of the sequence G1-G5.

4. DISCUSSION

5.1. Arithmetical problems

As to the differences between arithmetical and geometrical problems, it is worth remarking that in (complex) arithmetical problems verbal language is the main tool the pupil may use to design his strategy and to keep in touch with the problem situation. Drawings and diagrams may be useful, but they do not always succeed, specially with poor problem-solvers, as mediators between the meanings of the problem situation and the task of building a strategy. This may explain children's preference for strategies better characterized on the linguistic ground, like strategy A (§ 3.1.) where key words such as 'total' probably play a crucial role whether or not they are explicitly included in the text of the problem. This might also explain why the pupils who can use verbal language fluently in their reports in order to build a strategy are the only who answer correctly to either problem A4 and A5.

Moreover, the small number of shifts of strategies, and in particular the fact that no pupil wants to leave strategy A in the sequence A1-A5 once he has built it the first time, may depend also on the objective difficulty of understanding the equivalence of strategies. Most likely fifth-graders cannot yet be able to mastery the 'theorems in action' involved.

The predominance of strategy A in the sequence may also depend on its generality, as it is not affected by variations of the numerical data, whereas the other strategies may require a division in place of a multiplication according to the number of workers for any loom. Furthermore, it may be simpler for children, as it may be regarded as a combination of 'direct' operations (i.e. additions and multiplications) with only one subtraction as the last operation.

5.2. Geometrical problems

In geometrical problems, drawings allow to visualize a large amount of relations and make less urgent the need for verbal language in order to organize a procedure: a simple drawing may embody a complex procedure (such as the iterative construction of a figure) or explain that two figures have the same area. In other words, it can mediate between the meanings of the problem situations and the task of building a strategy. Some pupils, mainly in the first problems, use the drawings to check their conjectures (for example, by cutting or folding the paper or counting the squares) and then possibly decide to change strategy. In the last problems they seem to use more 'abstract' properties of the figures (e.g. equivalence).

A drawing, because of its 'ambiguity' [Laborde, 1988], allows the child to use his/her spatial intuition but, at the same time, proposes him/her some further questions which cannot be solved

only by intuition. A lot of pupils, at the beginning of the experiment, were able to report thei strategy as a sequence of operations with numbers, but not to describe their (corresponding operations on the figure. Their spatial intuition was an indistinct one, they did not know even th words to describe the figure in detail. At the end of the experiment, much of them were able t relate the operations on the figure with their numerical procedure, and to use words such a 'oblique side' 'height', 'equivalent'. The comparison of strategies often has related their spatia intuitions with the need/possibility of verbally expressing spatial properties in a more and mor precise way. All these properties of geometrical figures makes the 'theorems in action' involved i these problems much more evident and reliable for children.

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TEACHERS' CONCEPTIONS OF STUDENTS' MATHEMATICAL ERRORS AND CONCEIVED TREATMENT OF THEM

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This paper presents our first attempt to design an in-service teachers' training program for the treatment of students' errors and misconceptions in elementary school Mathematics. Thirty elementary school teachers were interviewed in depth on their conceptions about students' errors and treatment of them. Data for our analysis included teachers' answers on certain pilot-tested questions, their reactions to other hypothetical teachers' treatment of errors, and their own treatment of typical errors. Differences observed in the teachers' reported treatment of errors were related to differences observed in their conceptions of errors. However, there were instances where factors perceived by teachers as being out of their control seemed to influence their reported treatment of errors.

Introduction

Many researchers in Mathematics Education study students' errors in several mathematical topics based on different theoretical and methodological approaches. Some of these researchers mainly focus on the analysis of students' errors. They construct models that fit the conceptual operations leading to certain misconceptions (Hart, 1984; Steffe, 1988). Some other researchers also examine the role of instruction in overcoming these misconceptions (Onslov, 1990). Although many questions wait to be answered, some general recommendations for practice can be abstracted from this research. In general, the extent to which students can succeed in overcoming their errors depends on the teacher's efforts to devise challenging problem situations at an appropriate level of difficulty, and to create a classroom environment conducive to students' conflict discussions which will allow him/her to delve deeply into the students' underlying misconceptions leading to errors. The above actions are complex as they require from the teacher to move within the student's "zone of proximal development" and they become even more complex if the conventional school environment is taken into account. Thus, teachers need to be given support in their efforts for dealing with error while at the same time they must be prepared to accept doubt and uncertainty as unavoidable.

Teachers' support should be based on research data. Teachers' conceptions

step towards the design of effective support programs for them. For, as it has been argued in many researches, teachers' conceptions play such a significant role in their actual practices that they sometimes hinder the adoption and application of any innovative recommendations. The research reported in this paper is the first part of a program which has the following goals: (1) to study teachers' conceptions about students' mathematical errors and their conceived methods of treating them, (2) to examine the relationships between methods methods that teachers use in treating their students' errors in the classrooms and their conceptions of errors, (3) to understand the difficulties that teachers might experience when they try to implement the appropriate recommendations for dealing with errors in their classrooms, (4) to analyze the communication among students when they interact and strive to overcome errors, and (5) to investigate students' affective reactions towards their errors. In this paper we present our work on the first of the above goals.

Methodology

In this research thirty teachers with teaching experience ranging from 2 to 22 years participated. These teachers were teaching in elementary public and private schools of Athens where students coming from all the socio-economic levels are registered. Each teacher was interviewed for about one and a half hour. The interview consisted of three phases. In the first phase teachteachers were asked to answer a set of open-ended questions in order to reveal some aspects of their conceptions about errors and of their conceived methods of treating them. Observations of teachers while they were teaching and pilot interviews shaped this set of questions in its final form. During the second phase teachers were shown three transcripts of dialogues between hypothetical teachers and students and they were asked to comment on these teachers' treatment of errors. In the third phase of the interview, five errors that elementary school students usually make were presented to the teachers. They were asked to indicate their perceived causes of these errors and to suggest ways that they would use in dealing with them. In these last two phases we reduced the complexity of the decision-making process that teachers face when they come across students' errors in the classroom. In this way we thought that it would be possible to reveal the teachers'available store of methods for dealing with errors. All the interviews were audiotaped and the transcripts of them were analyzed in order to discover patterns in the teachers' responses.

Results

Results of our analysis on teachers' conceptions of errors and on their reported ways of treating them are presented in this section.

A. On teachers' conceptions of errors:In order to understand teachers' conceptions of errors teachers were asked to talk about the following: (1) the causes to which they attribute errors, (2) the consequences of errors on students' mathematical progress, (3) the different consequences that errors may have for the learning of Mathematics as compared to the learning of other subjects, and (4) their emotional reactions when students make errors. It was possible to classify teachers' responses on these topics into three distinct categories depending on the emphases placed by them. Each of them represents a different conceptualization of error in Mathematics. These conceptualizations are described below:

1. Nineteen teachers indicated that students' errors are mainly caused by inadequacies in their previous knowledge. In the examples that they mentioned these inadequacies were located in isolated concepts, facts, or procedures which students could not absorb. Furthermore, when these teachers referred to errors due to the students' inability to think, what they meant was that students were unable to apply already taught rules and tips for solving certain groups of problems. Family environment, textbooks, precipitation/carelessness, and inclination of students were other more general causes to which some of these teachers attributed errors. The consequences of errors were considered as negative for students' mathematical progress. Most of the teachers in this category noted that errors should be prevented rather than cured. As they indicated errors might impede the acquisition of new knowledge because they confuse students. In comparing the consequences of errors for mathematical learning to their consequences for the learning of other subjects, they asserted that there are differences due to the hierarchical nature of mathematical knowledge. They thus argued that mathematical errors are detrimental because they leave their traces and students may face difficulties in learning subsequent concepts. Most of the teachers in this category they that they experience feelings of anxiety and frustration towards their students' mathematical errors because these errors make them worried whether they have done a good job or not.

2. Eight teachers emphasized that students' errors could be attributed to their developmental level. These teachers viewed errors as a result of the students' cognitive unreadiness to approach certain mathematical concepts. In their examples they mentioned that students' deficiencies in classification, 1-1 correspondence, conservation and other logical concepts are determinant causes of errors. This view is also promoted in the teachers' guide textbooks which present some of Piaget's ideas. However, teachers could not provide adequate explanations about the relationship of these logical concepts to the explicit mathematical concepts and procedures that students misconceive. Other causes to which some of these teachers referred to, were family environment, age, textbooks and students' precipitation/carelessness. Teachers in this category viewed the consequences of errors as mainly negative. Most of them mentioned that errors may inhibit the students' natural development and thus subsequent learning. They also expressed their concern for the negative influence of errors on students' self-confidence. By reducing mathematics to logic these teachers thought that errors have in general more important consequences for the learning of Mathematics than for other subjects. In talking about their emotional reactions these teachers revealed that they feel frustrated. Some of them also mentioned that depending on the number of cognitively immature stu- in a classroom the intensity of this feeling seems to increase because of the

pressure to finish the book while many students are not developmentally ready to make sense of it.

3. Three teachers reported that students' mathematical errors are caused by the difficulties that students meet in approaching new knowledge. These difficulties as it can be inferred from their examples have their sources in the students' natural weaknesses to relate new knowledge to their informal experiences as well as to their already known Mathematics. Other more general causes mentioned by these teachers were the family environment of the students and the textbooks. The consequences of errors for students' mathematical progress were considered as being positive. These teachers expressed their concern for bringing to the surface their students' errors. For, as they said errors help students to experience conflict and thus to understand the inadequacies of their erroneous knowledge so that to reorganize it. No differences were noted between the consequences of errors in mathematical learning and in other subjects' learning. Finally, they felt challenged towards their students' mathematical errors and, unlike the other teachers, did not report any negative emotional reactions. It is interesting to note that their view as they confessed is not based on any awareness of the related research literature but it is formed by their reflection on their interactions with students and on their experiences as learners themselves.

B. On teachers' reported treatment of errors:the analysis of teachers' reported treatment of errors will be presented for each phase of the interview separately. This analysis revealed that teachers' responses could be classified into groups reflecting the groups in which their conceptualizations of errors were classified. However, as it can be noted in the following presentation there were some discrepancies between conceptions of errors and treatment of them referring to the teachers of the second conceptualization.

In the first phase of the interview teachers were asked to comment upon their perceived role in treating students' errors. For the teachers of the

first conceptualization it was revealed that, in correcting students' errors, they follow a systematic teacher-led procedure. When they are confronted by the students' errors they pose questions referred to previous concepts or facts that they themselves judge that the students do not know or do not remember. Thus, they lead students to the correct answer without giving them the chance to develop independent thinking. Students have to answer to these directed questions which usually admit only one correct answer. Otherwise, they have to listen passively to their class-mates answering these questions or to the explanations given by their teacher. These teachers also reported that they assign some similar exercises for extra-practice in order to ascertain that students will overcome their errors. Teachers of the second conceptualization tended to believe that their intervention could not be much effective. These teachers seemed to follow a similar to the procedure of practice described above. By further questioning them it was revealed that the general character of their practice was disapproved by some of them due to time constraints, the compulsory curriculum, the textbooks, and their training which were mentioned by them as factors influencing their practice. It should be noted, however, that they suggested more Piagetian type activities as means for overcoming students' errors. Teachers of the third conceptualization perceived their role in helping students to overcome their errors as particularly difficult. For, as they argued their students' way of thinking is sometimes quite different from their own. These teachers expressed their concern about having to involve actively all the students in the process of overcoming their errors through conflict discussions. However, they raised some questions related to the management of the classroom (i.e., "What can I do when a student offers a ready-made answer to the problem which one of his/her class-mates has?", "How can I keep the rest of the class busy when only a few students make an error?").

In the second phase of the interview three transcripts of teacher-student interactions were presented to the teachers and they were asked to comment on these hypothetical teachers' treatment of errors. These transcripts were designed in such a way so that to allow us to draw some conclusions about the teachers' views related to the amount of help that must be given to students who make an error. The majority of teachers did not seem to relate the amount of help given to students to the development of independent thinking. Only the teachers of the third conceptualization seemed to be aware of this relationship. However, they emphasized that a student must not be left at loss under any circumstances.

In the third phase of the interview teachers were asked to suggest specific ways that they would use in treating particular students' errors. Five typical errors were presented to them (i.e., one computational and four conceptual) and they were informed that these errors have been made by average ability students. Teachers of the first conceptualization did not appear to have a variety of methods at their disposal in correcting errors. Their reported methods were limited to the use of strategies, such as, repeat the teaching of the unit, key-words strategy, and summarizing the steps of an algorithm. Although teachers of the second conceptualization did not differ much in their reported strategies, they emphasized the use of some additional strategies like the use of concrete materials and the exemplary solution of problems simpler or similar to the ones that students have mistaken. Finally, teachers of the third conceptualization had a richer store of available methods. These methods included acting out the problem, make a drawing or a table, guess and test, and other strategies that students spontaneously use. It should be noted, however, that all of them expressed their discomfort because as they indicated they would like to know the particular needs of the students who made the errors. Moreover, they felt that, by interacting with them, it would be possible to determine the best treatment for their errors.

Discussion

The different conceptualizations of students' mathematical errors that teachers revealed in their responses were related to their reported treatment of errors. For the majority of teachers, errors are conceived in a negative way which, as it was apparent, it influences their reported ways of dealing with them. For a few teachers, there was not a strong relationship between their conceptions of errors and conceived ways of treating them. Factors, like time constraints, the compulsory curriculum, textbooks, and their training were reported by them as influencing their practices. Very few teachers seemed to have a dialectical relationship between their conceptions of errors and treatment of them.

Further research is needed in order to answer questions such as the following: (1) How do teachers with different conceptualizations about errors deal with them in their classrooms?, (2) What recommendations about the treatment of errors are appropriate for teachers of different conceptions of errors?, (3) Which difficulties teachers might experience in applying these recommendations in practice?, (4) Are teachers, who argued that the socio-economic level of the students' family influences the development of misconceptions, right?, (5) How can we answer the questions about the management of the class during conflict discussions which some teachers asked?, (6) How do teachers in other countries with different cultural and socio-economic conditions differ in their conceptualizations about errors and treatment of them?. Questions like the above are complex but, unless we answer them, the training of teachers will not be effective.

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CHILDREN'S UNDERSTANDING OF FRACTIONS AS EXPRESSIONS OF RELATIVE MAGNITUDE

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A study of 60 fifth graders revealed a strong tendency to identify the numerator as denoting literally the number of marked elements and the denominator as denoting the total number of elements. On the whole students did not accept as valid descriptors equivalent fractions which did not meet these criteria. Findings are discussed in terms of the concept of relative magnitude.

In the conceptual field of additive structures the relative magnitude of two numbers or quantities corresponds to the difference of the terms. One expresses this difference through statements such as "Ann has 3 marbles more than Tom". In the field of multiplicative structures, one encounters a different type of relative magnitude, namely, *relative magnitude in a multiplicative sense.* This concept assumes different forms in different representational systems. In natural language, one expresses the concept through sentences such as "Ann has 3 times the number of marbles Tom has". In written mathematical notation, it corresponds to a multiplicative operator. In a graph of a first order equation, it corresponds to the slope of the plotted line.

Relative magnitude bears directly upon the concepts of ratio and proportion. Although from the standpoint of the school curriculum these concepts become prominent shortly before adolescence, one can trace their psychological origins back to early childhood concepts associated with geometrical and spatial understanding and measurement--which, as Piaget, Inhelder, and Szeminska (1960) noted three decades ago, has "roots in perceptual activity (visual estimates of size, etc)". A recent investigation by Spinillo and Bryant (1991) found that children as young as 6 years of age show an emerging understanding of the equivalence of ratios in perceptual judgments of relative size.

The present study of fractions enters the discussion in a middle ground between fully geometrized problems (comparing non-arithmetized ratios of areas of different colors) and fully arithmetized problems (eg. solving the equivalence "3 is to 4 as 9 is to what?"): we will be dealing with problems of the equivalence a ratio of two numbers (expressed as fractions such as three-fourths) to a ratio of elements.

Fractions express the ratio or relative magnitude of a numerator to a denominator in the sense that neither of the terms refers to absolute amounts. For instance, if 1/8 of Mary's income is spent on clothing, we must not assume that she earns $8.00 (or 8 cruzeiros or whatever), $1.00 of which is spent on clothing. While this is clear to mathematics educators, it may not always be to children.

In looking over the data of Lesh, Landau, & Hamilton (1983), we noticed that 4th through 8th graders had high levels of success on problems in which the numerator of a fraction corresponded directly to the number of marked elements, and the denominator to the total number of elements illustrated. For example, 80% of the fourth graders and 98% of the eighth graders correctly solved the star problem (problem 1 in Appendix 1) drawn from Lesh et al. (1983). However, in problems where no such direct correspondence occurred success rates were much lower. For example, in the Lesh study, only 10% of the fourth graders and 38% of the eighth graders solved the triangles problem (problem 2 in Appendix 2).

Hart (1981) reports a use of counting in assigning fractional values to partitioned geometric figures, some parts of which are highlighted: "Questions of this type were almost invariably solved by counting the number of squares in the entire figure, counting those shaded and putting one number over the other" (p. 69). This is of course a correct way to approach the problems. But one wonders whether, when forced to decide if a particular fraction properly describes such figures, students will choose an equivalent fraction if the "literal" fraction is not present among the alternatives. It is important to note that this is not quite the same as asking whether children will solve "equivalent fractions" problems (Eg., find X if 2/3 = X/9) nor "reduce the fraction" problems, which are essentially problems of computation. The issue is rather whether children consider it legitimate to use, say, the fraction 10/24 to describe a figure of 5 eggs circled in a dozen. Children who think it is not legitimate are clearly treating the numerator and denominator in an absolute or literal rather than relative way.

The data in the Lesh et al. study suggest to us that many children were inclined to treat fractions in a literal sense. Our hypothesis is that many children got the first problem correct not because they truly understood fractions in a relative sense but rather because they simply counted and matched the number of highlighted cases and total number of cases to the numerator and denominator, respectively. However, since the problems used in the Lesh et al. study differed in several respects we cannot be certain regarding the source of the differences. For example, the triangles problem mentioned above contains a larger and more diverse set of elements than the stars problem. By experimentally manipulating the proper features, we can eliminate the role of extraneous features and put our thinking to test.

A total of 12 problems (included in Appendices 1 and 2) were used in the present study. Eight problems are derived from the Lesh et al. study. Six of these are described according to the following rationale. In the literal version of a problem, the numerator and denominator corresponded to the actual number of highlighted and total items. A relative version of the stars problem used the expression "four-sixths" in place of "two-thirds". A literal version of the triangles problem included the fraction 6/18 in the place of 1/3. A third problem showed 5 eggs circled in a carton of 12. The literal version has the answer 5/12 among the alternatives; the relative version includes the fraction 10/24.

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As an additional control, a fourth problem was adapted in which two correct answers--literal and relative--were presented among the alternatives so that we could look directly at students' preferred choice. The original problem in Lesh et al. had asked students to find one sixth of twelve. Since this expression sounds somewhat awkward in Portuguese, we dropped the "of twelve" and produced a problem asking simply for "one-sixth" (um sexto) and another asking for "two-twelfths" (dois doze avos).

We expected students to perform better on literal versions of problems and to opt for literal answers when both types were available. Underlying our reasoning was the suspicion that students were using a "literal fractions scheme" which entailed enumerating countable elements. If this were indeed the case, they should be puzzled when presented problems not amenable to solution through counting and matching.

Four additional problems were designed to investigate this possibility. The problems depart from fractions problems as normally encountered in school in at least three ways. First, quantities were represented by line lengths rather than areas. Second, the lines were assigned numeric values different from 1. Finally, the problems involved improper fractions. In the problems a complete line segment is assigned a value, such as 7 or 9. The line segment is already partitioned into equal parts. One part is highlighted. The student is asked to give the value of the highlighted part.

In one problem a line with a value of 7 is partitioned into 3 parts. Students are asked the value of what happens to be one third of the line. If students' thinking is heavily influenced by the part-whole model used in schools, in which the whole is assigned a value of 1, one would expect to find many students selecting 1/3 as the answer. It is indeed true that the highlighted part is "one third of the line", but if the value of the whole line is seven, hence not 1, then the correct value must be 7/3. Now, there are several ways of expressing 7/3. One is to call the expression "seven thirds"; another is to call it "one-third of seven". According to our reasoning, the first option can not be assimilated by a literal schema in which the numerator refers to the number of highlighted parts since there are not seven highlighted parts, but rather, only one. Furthermore, an improper fraction ought to be mysterious to students raised on a part-whole model in which the maximum possible number is one. An alternative in the second version of the problem (one third of seven) does not violate the schema and hence is expected to be relatively congenial to students' ways of thinking about fractions. The answer 1/3, which simply ignores the value 7, is also consistent with the schema.

A similar line of reasoning applies to the problems in which students are asked to give the value of one part (in four) of a line segment assigned the value of 9. The alternative, 9/4, should correspond to an illustration in which nine of something are marked off in a total of four! We thus expect only a very small percentage of students to select this answer. We were not quite sure how to predict student's reactions to the alternative "2¼". Our hunch was that it would be somewhat more favoured over 9/4.

Sixty fifth graders from a public school located within the campus of a Federal University in Recife, Brazil took part in the present study. All students answered both forms of the questions. Half solved form A problems and returned them to the examiners before obtaining form B; the other half answered form B questions first. In this way it was possible to control for order effects and to assure that students would not be able to redo problems solved earlier when given their counterparts. Any systematic differences in responding found between the two forms could thus not be attributed to the students' having seen similar problems earlier.

RESULTS

1. Literal versus relative problems.

The results are organized by problem types. A summary of the major effects is shown in Table 1 (below).

(a) The stars problem. Striking differences were found in the two versions of the stars problem. While fully 75% of the students accepted the expression "two thirds" to describe two stars marked among a total of three, only 33% of the students accepted the label, "four sixths". In the latter case 60% of the students choose the "not given" alternative; presumably they were looking for the answer "two thirds", not present among the alternatives.

(b) The triangles problem. Again, there was a large difference in success between the literal (50% correct) and the relative (27%) versions. A popular alternative was the fraction 6/6 (35% in the literal version; 50% in the relative version). This choice seems to indicate a reluctance against considering the total set of geometric figures, that is, against using diverse objects as referents for the denominator.

(c) The eggs problem. 90% of the students accepted 5/12 as a description of the illustration wherein 5 of 12 eggs were circled. Only 35% of the students accepted the label 10/24; in this relative version more than half of the students (55%) chose "not given". They were probably hoping to find the answer 5/12 among the alternatives.

2. Problems with both a literal and relative answer present.

In these problems (see the last two lines of Table 1) there was an overwhelming tendency to select the literal alternative when both were available.

(a) one sixth. 88% of the students selected the figure of one circle marked in a total of 6. Only 6% chose the figure of 2 balls marked among 12. Only one student gave two answers, both of which were correct.

(b) two twelfths. In this problem, what was formerly a literal response (one-sixth) was now a relative response, and vice-versa. Students continued to express their strong preference for literal answers: 88% of the students selected the figure of two circles marked in a total of 12. Only 6% chose the figure of 1 ball marked among 6. No students marked more than one answer.

Table 1: Selection of literal versus relative responses*.

	TYPE	
PROBLEM	Literal	Relative
Stars	75%	33%
Triangles	50%	27%
Eggs	90%	35%
"one-sixth"	88%	6%
"two-twelfths"	88%	6%

* For the final two problems the percentages refer to the tendency to select each type of answer, when both were available. Thus, for the first four problems each student (N=60) gave two answers; for the last two, they selected one answer, with the exception of one student, mentioned in the text, who gave two answers to the one-sixth problem.

3. Number line problems.

(a) Line with value of 7; one part in three is marked off. Students must determine its value.

Only 3% of the students accepted the correct expression, "seven thirds", to describe the value of the marked segment. Roughly half of the students (47%) preferred the answer "1/3". Another 17% chose the answer 3/7. In the other version of the problem, in which the correct expression "one third of seven" was available, 33% of the students chose it. Although this is an improvement over the first version, a yet greater percentage (60%) selected the alternative "not given". It is not clear what answer they were hoping to find among the alternatives.

(b) Line has value of 9; one part in four is marked off. Students must determine its value.

The most marked finding among the two questions involving the last problem is that neither of the two correct alternatives, 9/4 and 2$\frac{1}{4}$, tended to be accepted, with only 7% and 1% of the responses going to these options.

These last results suggest that there are subtle semantic issues which remain to be clarified. Since, as pointed out above, the questions differed in several ways from the problems students normally met, further research is needed. In any case, it is clear that to fifth graders the expression "one third of seven" appears to be very different in meaning from the expression "seven thirds". This issue would appear to relate to the the kinds of schemas children develop and use to make sense out of diverse situations involving mathematics.

CONCLUSIONS

The present study indicates that fifth graders are disinclined to treat the numerator and denominator of fractions in relative sense when applied to discrete elements. Most students will correctly assign fractional values to geometric figures provided that there is a direct correspondence between the number of elements marked off and the numerator, on one hand, and the total number of elements and the denominator, on the other. But when the number of elements counted does not match the numerator and denominator in this manner, they will refuse to accept the fraction as a valid label for the figure. It is not clear whether this is due to a misunderstanding on the part of pupils about what fractions are meant to refer to, encouraged perhaps through literal examples, or whether we are dealing essentially with a cognitive-developmental phenomenon. Although quite young children appear to have an intuitive understanding of relative magnitude in non-numerical perceptual judgment problems, how they develop an understanding of numbers as expressions of relative magnitude remains an important topic for research.

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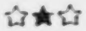
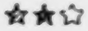
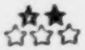
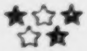
Appendix 1 - Form A problems.

#1. Which picture shows two-thirds shaded?

a. 

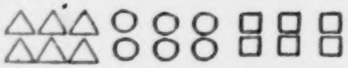
b. 

c. 

d. 

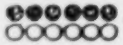
e) Not Given

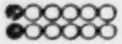
#2. What fraction of the set of objects are triangles?

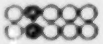


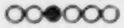
a) $\frac{6}{6}$ b) $\frac{1}{2}$ c) $\frac{2}{1}$ d) $\frac{6}{18}$ e) Not given

#3. Which picture shows one-sixth?

a. 

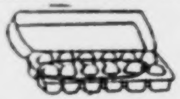
b. 

c. 

d. 

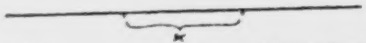
e) Not Given

#4. What fraction of the eggs are circled?



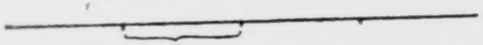
a) $\frac{18}{24}$ b) $\frac{5}{7}$ c) $\frac{5}{8}$ d) $\frac{7}{12}$ e) Not Given

#5. What is the value of the part labelled x if the whole line has the value 7?



a) $\frac{1}{3}$ b) $\frac{1}{7}$ c) one-third of seven d) $\frac{3}{7}$ e) Not given

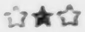
#6. What is the value of the part labelled x if the whole line has the value 9?

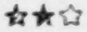


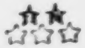
a) $\frac{1}{4}$ b) $\frac{9}{4}$ c) $\frac{1}{9}$ d) $\frac{4}{9}$ e) Not Given

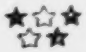
Appendix 2 - Form B problems.

#1. Which picture shows four-sixths shaded?

a. 

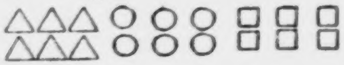
b. 

c. 

d. 

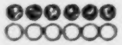
e) Not Given

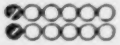
#2. What fraction of the set of objects are triangles?

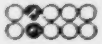


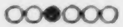
a) $\frac{6}{6}$ b) $\frac{1}{2}$ c) $\frac{1}{3}$ d) $\frac{2}{1}$ e) Not Given

#3. Which picture shows two-twelfths?

a. 

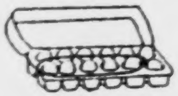
b. 

c. 

d. 

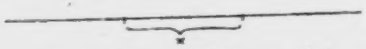
e) Not Given

#4. What fraction of the eggs are circled?



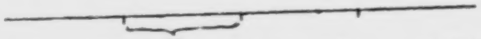
a) $\frac{5}{12}$ b) $\frac{7}{12}$ c) $\frac{5}{8}$ d) $\frac{7}{12}$ e) Not Given

#5. What is the value of the part labelled x if the whole line has the value 7?



a) seven-thirds b) $\frac{1}{3}$ c) $\frac{1}{7}$ d) $\frac{3}{7}$ e) Not given

#6. What is the value of the part labelled x if the whole line has the value 9?



a) $\frac{1}{4}$ b) $2\frac{1}{4}$ c) $\frac{1}{9}$ d) $\frac{4}{9}$ e) Not Given

DIFICULTAD EN PROBLEMAS DE ESTRUCTURA MULTIPLICATIVA DE COMPARACION

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SUMARY

In this paper a classification of multiplicative word "compare" problems is exposed and also the effect of two task variables on the difficulty levels of these types of problems is shown. The results were obtained in a paper and pencil test administered to 216 children's in 5th and 6th primary levels (10 and 11 pupils years old) and the data analysis procedure was the factorial and repeted measures ANOVA. Moreover we present the effect of school levels (5th or 6th) and the kind of question (assignment or relational sentences) on the observed achievement.

INTRODUCCION

Los conceptos, representaciones, problemas y procedimientos que surgen en relación con los problemas de multiplicación y división están recibiendo una atención aceleradamente creciente en los últimos años. La multiplicación y división de números, principalmente en los grados intermedios, se identifica como uno de los temas principales en la Agenda de Investigación denominada "Number Concepts and Operations in the Middle Grades" (J. Hiebert, 1988).

Aunque se han hecho estudios relevantes sobre problemas verbales multiplicativos (Bell, Fischbein y Greer, 1984; Fischbein, Deri, Nello y Marino, 1985; Schwartz, 1988; Vergnaud, 1983) aún "es necesaria más investigación para determinar la dificultad relativa de los diferentes tipos de problemas verbales multiplicativos" (Nesher, 1988, pp 39).

En la actualidad hay un acuerdo entre los investigadores sobre la existencia de tres grandes categorías de problemas multiplicativos, que denominamos "Problemas de aplicación de una Regla" (Mapping Rule), "Problemas de Comparación" y "Problemas de Producto Cartesiano"

Nuestro propósito en este trabajo consiste en delimitar variables que intervienen en los problemas multiplicativos de comparación elementales, discutir su relevancia, señalar dos variables estructurales destacadas y realizar un estudio piloto sobre el efecto de ambos factores en los niveles de dificultad observada en los diferentes tipos de problemas determinados de acuerdo con los dos parámetros

PROBLEMAS DE ESTRUCTURA MULTIPLICATIVA DE COMPARACION

Tipos de verbos

Los problemas que hemos estudiado son problemas verbales de comparación y, por tanto, son entidades textuales estáticas localizadas en un tiempo T_0; su enunciado no podrá expresar acción desarrollada en un tiempo. En los enunciados de nuestros problemas no intervienen verbos de acción, solamente verbos de existencia o estado.

Tipos de proposiciones

Siguiendo a Mayer (1985) distinguimos tres tipos de proposiciones en el enunciado de estos problemas: *asignaciones, relaciones y preguntas o interrogaciones*.

Proposición asignativa: atribuye o asigna un valor a una variable o cantidad. La *asignación* A consta de un estado E y de un valor numérico n atribuido a E.

Proposición relacional: expresa o establece una relación cuantitativa entre dos asignaciones. Para indicar que la relación R conecta las asignaciones A_1 y A_2 escribimos abreviadamente $R(A_1,A_2)$.

Proposición interrogativa : en nuestro contexto de investigación una proposición se llama interrogativa cuando pregunta o interroga sobre el valor numérico de una cantidad.

Tipos de magnitudes sobre las que vamos a trabajar

Imponemos dos limitaciones en nuestro estudio:

i) Las dos cantidades que aparecen en el enunciado de los problemas pertenecen a la misma magnitud y son cantidades de objetos discretos, por tanto, el tipo y número de magnitudes empleadas son variables controladas.

ii) Como las cantidades empleadas en los enunciados de los problemas son cantidades de objetos discretos los valores numéricos que le asignamos: m y n, son números naturales. Los números m y n empleados se hacen variar de forma controlada, manteniéndolos todos ellos inferiores a 100.

La Relación de Comparación Multiplicativa: Términos comparativos

La proposición relacional está determinada por los comparativos con los que viene expresada la comparación entre las dos asignaciones, a los que denominamos términos de comparación.

En el idioma castellano se utilizan tres tipos de términos comparativos, correspondientes a los tres tipos de comparaciones siguientes: *comparación de superioridad* que se forma incluyendo la expresión "más...que"; *comparación de igualdad* que se forma con "tanto...como" o "tan...como", y la *comparación de inferioridad* que se forma con "menos...que".

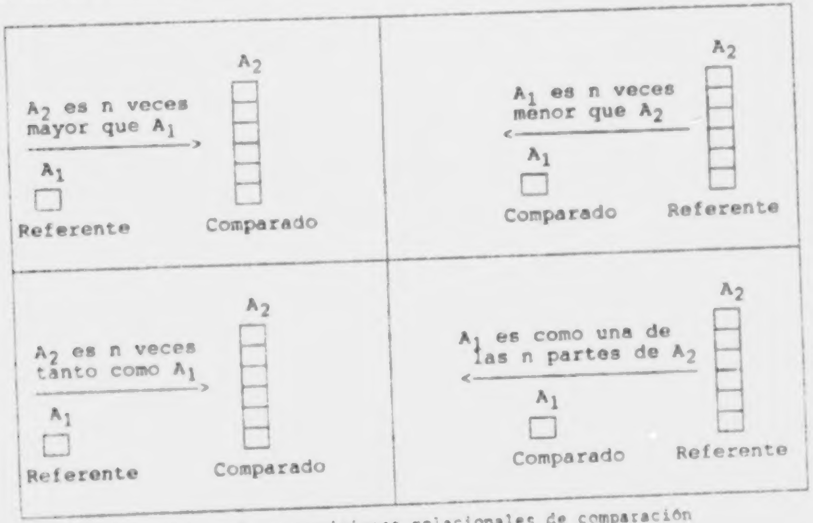
En los problemas multiplicativos de comparación se añade el término "veces", que dota al comparativo de su carácter multiplicativo

Las comparaciones que se hacen en base "veces más que" y "veces menos que" son claramente inversas una de otra. En la relación "veces más que" se expresa que el conjunto mayor contiene un número de veces una parte equivalente al conjunto menor, por tanto *la inversa debe expresar la idea de que el conjunto menor es como una de las partes iguales en que se puede descomponer o está descompuesto el conjunto mayor.* De acuerdo con esto, en su forma más general, la relación inversa lleva implícita la partición del conjunto mayor en partes iguales, es decir, fraccionar el conjunto mayor, o el que esa partición ya esté hecha.

Tenemos así la relación inversa buscada que tiene sentido en castellano utilizando el término comparativo "una parte como".

La proposición relacional: factor R

Fijadas las condiciones anteriores surgen cuatro tipos de relaciones de comparación multiplicativa que aparecen en el cuadro siguiente:


A_2 es n veces mayor que A_1
A_1
A_2
Referente
Comparado
A_1 es n veces menor que A_2
A_2
A_1
Comparado
Referente
A_2 es n veces tanto como A_1
A_2
A_1
Referente
Comparado
A_1 es como una de las n partes de A_2
A_2
A_1
Comparado
Referente

Cuadro 1 Tipos de proposiciones relacionales de comparación

En cada caso hay una proposición relacional sintácticamente distinta, y hacemos referencia a cada una de ellas por la expresión relacional que la determina, a la que nos referimos también como "el término relacional" o "término de comparación". Esta variable, que hace referencia al tipo de proposición relacional y que toma cuatro valores, la llamamos factor R.

Desde el punto de vista funcional estos problemas están caracterizados por incluir en su enunciado dos funciones proposicionales de una variable: $A_1(x)$ y $A_2(y)$ y una función proposicional relacional de las variables anteriores: $R(A_1,A_2)$. Un enunciado concreto de un problema de comparación multiplicativa surge de particularizar o asignar un valor numérico a dos de estas funciones y solicitar que se halle el valor de la

tercera función. Entra en juego así otra variable, que es la posición de la incógnita.

Posición de la incógnita: factor P

En el caso de los problemas de estructura multiplicativa que estamos considerando la variable y s obtiene de componer la variable x con el factor de comparación al que podemos llamar α, es decir, podem escribir la relación cuantitativa que existe entre las variables en la forma $y = \alpha \cdot x$. Los tres tipos de probl mas citados surgen de considerar desconocida algúna de las variables que intervienen en esta ley, en concreto:

$$y = \alpha \cdot ? \qquad y = ? \cdot x \qquad ? = \alpha \cdot x$$

Estas expresiones se conocen con el nombre de *sentencias abiertas* y, como se puede ver, es la *posición de l incógnita* lo que cambia de una sentencia a otra.

Los doce problemas de comparación que estudiamos surgen de considerar simultáneamente el factor R co cuatro valores, y el factor P con tres valores. Un ejemplo de cada tipo puede verse en el cuadro 2.

Tipo (R_1,P_1): 1.-Daniel tiene 12 canicas María tiene 6 veces más canicas que Daniel ¿Cuántas canicas tiene María?	Tipo (R_3,P_1): 7.-Daniel tiene 12 canicas. María tiene 6 veces tantas canicas como Daniel. ¿Cuántas canicas tiene María?.
Tipo (R_1,P_2): 2.-Daniel tiene 12 canicas María tiene 72 canicas ¿Cuántas veces tiene María más canicas que Daniel?	Tipo (R_3,P_2): 8.-María tiene 72 canicas. Daniel tiene 12 canicas. ¿Cuántas veces tiene María tantas canicas como tiene Daniel?.
Tipo (R_1,P_3): 3.-María tiene 72 canicas María tiene 6 veces más canicas que Daniel ¿Cuántas canicas tiene Daniel?	Tipo (R_3,P_3): 9.-María tiene 72 canicas. María tiene 6 veces tantas canicas como Daniel. ¿Cuántas canicas tiene Daniel?.
Tipo (R_2,P_1): 4.-Daniel tiene 12 canicas. Daniel tiene 6 veces menos canicas que María. ¿Cuántas canicas tiene María?.	Tipo (R_4,P_1): 10.-Daniel tiene 12 canicas. Daniel tiene tantas canicas como una de las 6 partes que tiene María ¿Cuántas canicas tiene María?.
Tipo (R_2,P_2): 5.-Daniel tiene 12 canicas. María tiene 72 canicas. ¿Cuántas veces menos tiene Daniel las canicas que tiene María?	Tipo (R_4,P_2): 11.-Daniel tiene 12 canicas. María tiene 72 canicas. ¿Qué parte son las canicas de Daniel comparadas con las de María?.
Tipo (R_2,P_3): 6.-María tiene 72 canicas. Daniel tiene 6 veces menos canicas que María. ¿Cuántas canicas tiene Daniel?.	Tipo (R_4,P_3): 12.-María tiene 72 canicas. Jaime tiene tantas canicas como una de las seis partes que tiene María. ¿Cuántas canicas tiene Daniel?.

Cuadro 2 Clases de problemas verbales de comparación multiplicativa.

ESTUDIO REALIZADO

Objeto del estudio

Esta investigación perseguía esclarecer cuatro puntos fundamentales referidos a los 12 tipos de problemas de estructura multiplicativa de comparación:

1º) Detectar en su resolución posibles diferencias de rendimiento entre alumnos de dos niveles: 5º y 6º de enseñanza primaria (niños de 10 y 11 años).

2º) Confirmar si los problemas cuya pregunta es una interrogación relacional son más difíciles, en términos de porcentajes de éxitos, que aquellos cuya pregunta es una interrogación asignativa.

3º) Determinar si las variables R y P influyen significativamente sobre la dificultad de resolución de los problemas de comparación multiplicativa.

4º) Estudiar un posible efecto de interacción entre las variables R y P.

Método

Para evitar el efecto de aprendizaje, los doce problemas de comparación multiplicativa fueron distribuidos en tres test homogéneos de 4 items cada uno. En cada test los items se distinguen por ser problemas que incorporan en su enunciado distinto término de comparación.

Los tres primeros items de cada test se diferencian también por la distinta posición de la incógnita. En el cuarto item de cada test la incógnita ocupa la misma posición que en el primer item.

En los tres test los números y los contextos empleados fueron los mismos. También se controlaron las variables sintácticas involucradas en los enunciados de los problemas.

Los tres test de lápiz y papel se aplicaron a ocho grupos de 27 niños pertenecientes a 4 colegios. En cada colegio se eligió un grupo de 5º y otro de 6º de enseñanza primaria. Dos de estos colegios están situados en el centro y los otros dos en la periferia de la ciudad de Granada. Cada alumno respondió a un test, cuya distribución se realizó al azar con arranque aleatorio.

A los alumnos se les pidió que escribieran a continuación del enunciado de cada problema las operaciones necesarias para obtener la respuesta. En el caso de que para un problema el proceso seguido por un alumno fuera correcto se da como buena su solución.

Resultados

Hemos utilizado dos ANOVA diferenciados para el análisis de los datos. En el primero, realizamos un ANOVA de la puntuación total obtenida por cada alumno en función de las variables COLEGIO, CURSO y TEST. En este primer análisis no hemos hallado diferencias significativas debidas al colegio o al tipo de test.

Sí se ha hallado una diferencia significativa entre cursos (F = 10.311; p = 0.002).

No hemos hallado interacciones significativas entre colegio, curso y test.

En el segundo ANOVA realizado, la puntuación de cada niño en cada problema se contrastó en función de los factores R y P. Los resultados obtenidos fueron:

a) No hemos hallado diferencias significativas debidas al factor R (F = 2.30; p = 0.07).

b) Hemos hallado diferencias significativas debidas al factor P (F = 39.52; p = 0.000).

c) Se ha observado un efecto de interacción entre las variables R y P, como puede observarse en la Tabla 1. En cada casilla de esta tabla se han incluido dos datos: la proporción de respuestas correctas para la categoría de problema correspondiente y el intervalo de confianza al 95 % colocados según la siguiente disposición:

Primer dato: proporción de respuestas correctas en los items correspondientes al valor dado de la combinación de factores.

Segundo dato: intervalo de confianza al 95 %, calculado a partir de los resultados de los programas de análisis de la varianza.

Al comparar las proporciones que aparecen en la Tabla 1 en base a los intervalos de confianza calculados observamos que las proporciones de éxitos para la variable P no es idéntica respecto a cada valor de R. Es decir, observamos la presencia de interacción.

Estos resultados se discuten con detalle en Batanero y otros (1991).

	R_1	R_2	R_3	R_4	QUINTO	SEXTO	TOTAL
P_1	.85 .826-.874	.56 .529-.591	.83 .803-.857	.58 .549-.611	.657 .571-.744	.745 .672-.819	.701 .643-.758
P_2	.26 .236-.284	.31 .289-.341	.46 .433-.487	.44 .409-.471	.273 .191-.355	.472 .384-.561	.373 .312-.434
P_3	.63 .606-.654	.60 .569-.681	.36 .333-.387	.74 .709-.771	.495 .404-.587	.597 .507-.687	.546 .482-.610
QUINTO	.491 .395-.587	.389 .295-.482	.528 .432-.623	.528 .432-.623			
SEXTO	.667 .576-.757	.583 .489-.678	.574 .479-.669	.648 .557-.740			
TOTAL	.579 .512-.645	.486 .419-.553	.551 .484-.618	.588 .522-.654			

Tabla 1: Resumen de medias e intervalos de confianza obtenidos para los doce tipos de problemas.

CONCLUSIONES

De los resultados obtenidos en esta investigación se desprende que es posible encontrar subcategorías de problemas dentro de la clase de problemas verbales de comparación multiplicativa. Estas subcategorías presentan diferentes niveles de dificultad para los escolares en los niveles entrevistados.

En primer lugar las diferencias observadas entre problemas asignativos y problemas relacionales es lo suficientemente importante como para aceptar que los problemas relacionales que hemos estudiado presentan un nivel mayor de dificultad que los asignativos, para la muestra de escolares tomada.

Puesto que los problemas estudiados son de dos tipos distintos de dificultad según la posición de la incógnita, el análisis del factor P nos lleva a confirmar que no sólo hemos encontrado diferencias significativas entre problemas asignativos y relacionales, sino que además dentro de los asignativos podemos distinguir dos tipos según la posición de la incógnita: P_1 y P_3.

El análisis efectuado para el factor R nos lleva a admitir que este factor ha tenido menos influencia sobre la dificultad de resolución que el factor P.

El factor R considerado aisladamente no ha presentado un efecto significativo sobre la dificultad de los problemas de comparación para los niveles entrevistados, pero al combinarlo con el factor P produce un efecto de interacción que sí es significativo, por lo que es necesario tener en cuenta a ambos factores simultáneamente para categorizar este tipo de problemas en los niveles considerados.

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CONSTRUCTION AND INTERPRETATION OF ALGEBRAIC MODELS

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The research illustrated in this report concerns two kinds of teaching situations that are often utilised for the approach to algebraic models in problem solving. "Short-hand" aspects of formulas construction prevail in the first situation, whilst in the second formulas transformation is necessary. The difficulties and obstacles encountered by the pupils are analysed and compared on the basis of a common interpretative scheme. Some open questions, of an educational or cognitive interest, are also indicated.

1. Subject, context and methodology of the research

In this research we propose to examine and understand some of the difficulties and obstacles concerning the approach to algebraic models and their command in problem solving situations. We will limit our examination to two types of problem situations that may be used to introduce algebra and algebraic models. The first kind, very frequently used in teaching, is characterised a priori by the presence of elements that may easily lead the pupil to construct and interpret algebraic formulas by means of a shorthand process and by not being required to transform the formulas straightaway. For these kinds of problem situations we will consider, in particular, problems in which the objective is to calculate one or more numerical values dependent upon variables and parameters which the pupils must point out and relate each to the other ones. In these cases, the algebraic model comprises one or more formulas in which the letters may be subsequently substituted by particular numerical values.

The second type, on the other hand, is characterised by the fewer stimuli towards formulas construction "short-hand" processes and by the intrinsic need to transform the formulas produced. For this type of problem situations we will consider, in particular, problems which may be configured as simple proofs of theorems given verbally to the pupils. In these cases, in general, it comes down to writing an algebraic model and to manipulate it according to the rules of algebraic calculations until a model is attained from which the pupil is able to deduce the truth or falseness of the proposition in question.

In paragraph 2 we will propose a common interpretative scheme of the difficulties and obstacles which arise in the solution of the problems considered. This scheme is based upon an "a priori" analysis of the abilities and attitudes which intervene in the construction and interpretation of the algebraic models in problem solving activities.

In paragraphs 3 and 4 we will examine, in accordance with the introduced scheme, the behaviour of the pupils in some of the first and second type situations described above.

In paragraph 5 we will draw some conclusions concerning a comparison between the two types of situations in relation to their development and handling in the classroom. Finally in paragraph 6 we will briefly refer to some open questions regarding the cognitive behaviour of the pupils while they are attempting to construct an algebraic model for solving a problem.

Our research falls within the context of a curricular project for comprehensive schools, concerning the teaching of mathematics and science. The project also provides for the use of the computer in problem solving. Within this framework the difficulties to construct, interpret and recall algebraic models appear to constitute one of the impediments to the transposition of resolution strategies into statements sequences and to the interpretation of single statements or sequence of statements written by others. For the last ten years the project has involved about 2000 pupils every year. The analyses reported in this paper, however, has involved a small number of pupils (about 200) from 13 classes in the last three years. It was preferred to refer only to those teachers who assured us that they would not intervene in the work carried out by the pupils during given tests, and who would systematically teach giving attention to aspects of recording and representing the adopted solution strategies. This second condition is important because our research work is based almost entirely on the written work produced by the pupils during their classwork. Only some interpretative assumptions which we shall make, in addition to the written work, also refer to teacher-pupil interviews performed upon completion of the written work.

2. An "a priori" analysis of some abilities and attitudes connected with the command of algebraic models

With reference to the problems indicated in paragraph 1, it appears that we could distinguish between abilities and cognitive attitudes which intervene mainly in the construction and other ones, which intervene mainly in the interpretation of algebraic models.
As far as the construction of models are concerned, the pupils may find, depending upon the problem, the need to put in practice several of the following abilities:

- to resort to the algebraic language in the shorthand function and/or in that of transformation;
- to develop a space-time mental environment in which, on the one hand, the succession of the elementary operations necessary to write the single parts of the model is constructed step-by-step and, on the other hand, the correctness of the relations between the different parts of the model are organised and verified;
- to handle a continuous dialectic between the two mental attitudes in relation to the model to construct and to its parts: it means passing continually from an "internal location" into the single part of the model in question to an "external location", in order to keep under control the relations existing between the parts of the model and vice versa;
- to recall models already employed and memorised (as a whole, as an "object"), making them functional in respect to the correct solution of the problem in question; this requires that a recalled model, in addition to being pertinent, is re-used by interfacing correctly the variables contained with those of the model which is being constructed.

- to exercise an awareness of the fact that to construct a model does not mean finding a strategy to solve the problem in question, but rather to determine a solution algorithm for a certain class of problems;
- to command not only the descriptive aspect of an algebraic model, as a representative model of a class of similar situations, considered implicitly as already existing, but also its previsional aspect as a generative model of future new possible situations.

It should be observed that in order to be able to write an algebraic model to solve a given problem, this does not "per se" imply to have attained these last two levels of command.

As far as the activities of interpreting algebraic models are concerned we refer only to teaching situations in which pupils are somewhat aware of the significance of the variables used; for example, the pupils are required to compare different models written by themselves for the solution to a particular problem. In these situations the pupils must put into practice some of the following abilities:

- to know how to read the algebraic formula(s) making up the model, both determining the relations existing between the different variables involved and the procedure(s) which operate on the variables;
- in the comparison of different algebraic models, as possible models of a particular situation, to admit that a particular situation may be described with "apparently" different, but semantically equivalent, models corresponding to different manners of viewing the same situation;
- to acknowledge, where it exists, the equivalence of two algebraic models, in order to transform one into the other on the basis of the rules of algebraic calculation, breaking completely away from the thought strategies which may have produced the models.

3. Teaching Situations in the Approach to Algebraic Models in which the shorthand function of the algebraic language prevails

We will examine how pupils behave when facing frequent problems: three problems related to a geometrical context and one associated to the so called real problems.

Problem A (12 year olds):
Write a formula which enables you to calculate the area of the figure in Fig. 1.
Problem B (12 year olds):
Draw a rectangle of height "h" and base twice the height. Write two formulas which enable you to calculate the perimeter and the area of the rectangle respectively.
Problem C (12 year olds):
Write a formula which enables you to calculate the area of the dotted figure in the Fig. 2 Use the formula which you have written to calculate the required area.

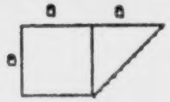

fig.1

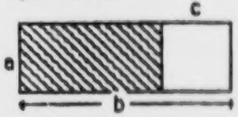

fig.2

Problem D (13 year olds):
A shopkeeper purchases a stock of 300 items of a certain household product costing Lire 1100 each, and spends a total of Lire $1100*300=330000$.
He then resells each item at a price of Lire 1800.
Hence if he sells n items he earns Lire $1800*n$ making a profit

$$G = \text{sales-cost} = 1800*n - 330000 \text{ Lire}$$

For example, if he sells 250 items, i.e. $n=250$, his profit amounts to:

$$G = 1800*250-33000 = 450000-330000 = 120000 \text{ Lire}$$

Use a simple pocket calculator to complete the table (n,G), assigning the values 50, 100, 150, 200, 250, 300 to n.

As far as Problem A is concerned the pupils essentially draw up two kinds of solution.
Solution 1: Area $= a^2 + a^2/2$; *Solution 2: Area* $= 3.a^2/2$.
Both when constructing one's own model and, subsequently, when comparing the various solution models the pupils correctly recall the algebraic models for the areas of squares and triangles. They have no difficulty to see the correctness of the the two formulas, because they interpret them by referring to the geometrical aid and, in the first case, they see a "square+triangle" and, in the second case, "3 equal triangles". During the comparison phase, however, we have noted that the pupils do not autonomously view any problem as the "equivalence" of the two formulas from a formal point of view.

As far as Problem B is concerned, the pupils elaborated six different solutions of the following kind (apart of the many incorrect solutions produced):
Solution 1: perimeter $= 2h+2h+h+h$; *Solution 2: perimeter* $= 2h+h+2h+h$
Solution 3: perimeter $= h*2+2h*2$; *Solution 4: perimeter* $= (h+h+h)*2$
Solution 5: perimeter $= 2h*3$; *Solution 6: perimeter* $= 6h$
Solution 6 is not very often used. In all of the other solutions it is clear that it is the geometric aid which provides a step-by-step guide in the construction of the formulas. The letter appears more to be a stereotyped name of the segment "height" rather than the name of a variable. The equations appear to mainly indicate the sequence of operations that the pupils perform from the drawn base and height segments for which they express the relation of "twice" in two ways (2h, h+h). The fifth equation generally accompanies the description of the kind "2h is the base, if multiplied by two we will determine the length of the two bases. Since the length of the base is twice the height, the two heights together form a third base, hence I have multiplied the length of the base by 3".

As far as Problem C is concerned the pupils produce the following five kinds of solutions:
Solution 1: Area $= axb - axc$; *Solution 2: Area* $= (b-c)xa$
Solution 3: Area $= axb$; *Solution 4: Area* $= axb - cxc$; *Solution 5: Area* $= bxa - c$
With reference to the graphics aid provided by the teacher and to the letters shown, solution 3 should be considered wrong. The fact, however, that the pupils who furnished this solution

then provided the numerically correct result to the question of calculating the area, has led us to analyse their algebraic model in a different manner. We believe, in fact, that the pupils recall an algebraic model concerning "area of the rectangle: axb, a being the base and b the height", without considering that the variables of the model used should be interfaced with those contained in the text of the problem (fig.2).
Solution 4 is strongly supported by the drawing which appears to suggest several pupils to consider the equality of the lengths of the sides (a,c) present in the drawing.

Finally we believe that in geometrical type problem situations such as that now presented, in the solutions prepared by the pupils, very little is knowingly expressed in an algebraic language and there is a very poor generalised level to problem classes (4).

As far as Problem D is concerned, the pupils do not generally have difficulty to construct the table and therefore to formally understand the variable "n". They are able, that is, to internally adhere to the given model. They are able to autonomously think, while they are filling in the table or with the encouragement of the teacher, of values of G as profits gradually accrued by the shopkeeper. We have, on the other hand, realised, through interviews or written questions after the given problem have been solved, that the pupils are unable to interpret the model as a previsional model, in spite of the fact that careful reading of the text of the problem should rightly suggest this view.
The children do not know how to reason in terms of "if the shopkeeper should sell .....". They are only able to think in terms of "when the shopkeeper has sold ....". To be able to give a provisional interpretation the children should be able to think of time as fixed and to organise in their mind a space in which to set a game of assumptions in order to reconstruct and highlight possible alternatives. All this involves considerable difficulties connected also to the manner in which this problem (or similar problems) are formulated and then solved by the children.

4. Teaching Situations for the Approach to Algebraic models, in which the algebraic language transformation and generalisation functions prevail.

In this paragraph we will analyse some pupil behaviours in teaching situations which appear to force the construction and the transformation of algebraic models as the only way to solve problems which by nature they question in general terms.

Problem E: *Show that the sum of two consecutive odd numbers yields a multiple of 4.*
Problem F: *Show that the result of the sum of two odd numbers is an even number.*
Problem G: *Show that the sum of two numbers multiple of 3 is still a multiple of 3.*

Before these problems, in which an immediate foothold to the shorthand function of the algebraic language is missing or insufficient, many pupils stop immediately or after having attempted to employ arithmetic techniques (for example: "3+6=9, 6+9=15, 9+15=24, 5+18=33... It is perhaps true."). With the difficulties which determine this stop aside, let us examine the behaviour of the pupils who commence with an algebraic approach.

In general, the pupils do not have difficulty to recall the model "algebraic sum of ...." in order to relate the sum of two variables nor to recall the models "2k+1" (odd numbers) and "3k" (number multiple of 3) respectively, used previously. They do not have any particular difficulty either to transform the models that they may have constructed, because the need for this transformation is contained in the text of the problem. It is the text itself that also indicates as to the kind of handling to be performed. The difficulties arise from the correct interfacing of the variables contained in the recalled model with those gradually input into the model under construction.

From this point of view, Problem E differs from Problems F and G in terms of the difficulty and expertise it requires. In problem E the pupils recall the "odd number" model a second time. From this they begin to construct the model of the consecutive odd number, either by arithmetic calculation (2k+1+2) or by algebraic analogy (2k+3). They complete the "algebraic sum of ..." model, and transform it into "4k+4" without any danger of making mistakes even if they disregard every questions regarding the interface of variable "k" used twice (in this case, in fact, "k" indicates only one variable, that is used to represent the first general odd number).

In problems F and G the pupils had considerable difficulty: they now found themselves recalling the same "odd number" or "multiple of 3" model a second time. However, in completing the "algebraic sum of ..." model the problem of interfacing the variables is fundamental. In fact, it concerns explaining, algebraically, that the two odd numbers or the two multiples of 3 being referred to in the two successive moments are, along general lines, determined by different values of variabie "k" of the respective models. The interfacing operation therefore requires that the names of the variables in the construction of the "algebraic sum of ..." model are distinguished. Otherwise the children find themselves showing that the sum of two equal odd numbers or two equal multiples of 3 is a number which has properties requiring proving and they become convinced to have found a general proof. They in fact conclude that 4k+2 (as the algebraic sum of the calculation of the expression "2k+1+2k+1") is an even number and that 6k (as the result of the algebraic calculation of the expression "3k+3k") is a multiple of 3.

What is interesting to note is that with respect to this kind of mistake there are no significant differences in relation to the age of the pupils, be they 11, 13 or 18 year olds in their first year at University.

Some Conclusions

The problem situations of the first kind seem much more accessible for the pupils, but nonetheless they often present the risk of an excessive emphasis on the shorthand function of the algebraic language due also to the fact that the teachers, when correcting the work produced by the pupils, often limit their actions to only checking the correctness of the formulas, finding confirmation in the graphic aid or in the calculation with numbers. On the other hand, we believe, that the formulas produced by the pupils could be very useful, if handled properly by the teacher, for comparison and thought activities in the classroom aimed at highlighting certain critical "aspects" of the shorthand process which has been used to generate them, and to also encourage activities of self-analysis in the approach to algebraic expressions. The models produced by the pupils could, moreover, become the basis of work to develop the transformation function of the algebraic language with goals understandable by the pupils such as, easier calculations and the verification of the equivalence between formulas.

As far as the situations of the second type are concerned, much more significant with respect to the transformation and generalisation functions, we cannot nonetheless underestimate the risk of a stop that they can cause in many pupils. On the basis of our experiences, we believe that we are able to assume that such a stop is related to the handling of classwork rather than to the age of the pupils. From an analysis of the work produced by the pupils it is possible to find, for instance, algebraic approaches in all pupils of a class or in none, irrespectively of the age (11, 12, 13 or 14 year olds). Algebraic and arithmetic approaches can be found in 18 year old youths coming from various schools, in the first few months of study at University.

6. Some open questions on two different behaviours in the construction of algebraic models.

Bearing in mind the state-of-art about procedural and relational thinking of pupils while solving problems, in our experiences we focused our attention on these two different behaviours ((3), (6), (7), (8)). The majority of pupils attempt to organise their own problem solving strategy, starting from a simplified case and finding, step-by-step, the operations to perform on data, perhaps after having pointed out some relationship between the data. Even in simple problems as Problem B in Paragraph 3, it is easy to clearly see, in the formulas produced, the part thereof lived and written in a procedural manner and the part in which the aspect of relationship between data dominates. In the equations furnished by the pupils there are many different ways of expressing multiplying processes: "2h", "2h*2", "h*2". In the first case the pupil points out the relationship between the two geometrical quantities which she or he sees. In the other cases the pupil expresses how he works on these quantities. In the case of more complex problems (Problem D of paragraph 3 or other in (2)), the solution strategy seems to be constructed in a mental space-time in which, according to the "time" component, the sequence of the operations to perform (experiencing the solution process from "within") seems to be organised and, according to the "space" component, the management of the choices and possible alternatives to manage (experiencing the process "externally") seems to take place. The discovery of the meaning and of the potential of the algebraic language emerges during this stage of the work, in which we feel that the teacher can perform a significant role to encourage the game of new assumptions, which lead from single cases to the general analysis of all possible cases. The final algebraic model which the pupils attain in this way describes, in algebraic notation, the entire dynamics of the operations on the data performed by the pupil and therefore acquires a meaning of an "a posteriori procedure".

Very seldom we find 13-14 year old pupils who attain the model by means of a global approach producing models as "a priori functional relation". These pupils do not appear to start from a particular situation around which to construct a solution strategy to generalise subsequently. They appear however, to show good command of the context in which the problem situation is set, attempting directly to express, in one or more formulas, the relations among the data of the problem. They no longer appear to reason in terms of the elementary operations to carry out in order to attain a solution, but rather to concentrate their attention on how to express the relationship between data, reasoning in operation "blocks" and for the meanings associated to such blocks ("models" recalled to be interfaced with the particular new problem situation). Most probably the game of assumptions made by these pupils leaves the traces in the protocols less visible. We believe, however, that it is important and that it is activated by reasoning of the kind "I believe that the most important things are.... ". In this case the command of natural language is just as important, both when understanding the text and when preparing the relationships between elements determined as being important. The complete construction of the final model may still require subsequent refinements, but they are of another kind in respect to those made by who follows the "a posteriori procedure" approach. These refinements are more of the kind "I think that we should also consider ....".

According to our experiences we feel that these two kinds of approach may be encountered also in the interpretation of given models, referred to problem situations presented explicitly also verbally. Reasons of space unable us to discuss this argument further.

Problems which require further analyses are the transition and the interaction between these two approaches, on condition that we are able to give an answer to the question whether these two approaches correspond to cognitive styles of each single pupil or if we should consider the one as the overcoming of the other (the production of models as "a priori functional relation", we feel, reveals the capacity to see globally all the possible situations, past and future) or if it is the complexity of the problem which pushes the pupils each time .

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ANALYSIS OF THE BEHAVIOUR OF MATHEMATICS TEACHERS IN PROBLEM SOLVING SITUATIONS WITH THE COMPUTER.

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ABSTRACT

In the framework of a computer science course for compulsory school mathematics teachers, their behaviour in problem solving situations are analysed through the comparison of solution strategies developed in three different computer science environments. These behaviours are related to the conceptions which they have developed of particular devices made available each time and to the cognitive processes to which they are subjected. The analysis carried out is set within our research programme aimed at examining, within the scope of computer aided problem solving the conditioning effects of the teacher's conceptions on the development of learning processes of the pupils.

Introduction

In this paper we intend to analyse some aspects of an experience on computer science training of compulsory school (11-14 year olds) mathematics teachers accomplished by us within the "Research Group on the teaching of mathematics and scientific education ", in which we operate as researchers.

The analysis concerns the behaviour of teachers in mathematical problem solving with the computer. In paragraph 1 we will describe the context in which the course was run and we will illustrate several principals which guided us in the design and running of the course. In paragraph 2 we will attempt to understand some of the behaviours of the teachers that emerged in problem solving situations encountered. In paragraph 3 we will attempt to discuss possible teaching and research implications resulting from the analysis performed in paragraph 2.

1. Context and principals guiding the planning and running of the course

The 25 teachers involved in the training course have, for many years, performed their activity as experimenters within our research group. They all have degrees in a wide range of fields (mathematics, chemistry, biology, ...). They have acquired extensive experience in the teaching of mathematics and experimental sciences. They have developed expertise in analysing the thought processes of the pupils in their learning of the various subjects.

On the other hand, we can define their computer science expertise, possessed at the outset of the course, as being "of first literacy". They had sufficient knowledge to introduce programming with 13-14 year old pupils, working with programmable pocket calculators.

The training activity was characterized by working on problems chosen in such a way that the quantity of data involved and their organisation have an important role in the construction of a solution strategy.

The problems had to be solved within three different computer-based environments (first in Basic, then in Pascal and finally in the spreadsheet "Excel" for Macintosh environments), because we feel that the comparison between the various cognitive processes involved in the construction of a solution strategy, on account of the three environments, could have permitted:

- to understand not only the hurdles and the difficulties which are present when designing an efficient solution strategy, according to a particular available environment, but also those that emerge when it is necessary to ignore the strategy achieved in a certain environment in order to be able to design another in relation to a new environment;
- to understand the importance and the need to develop a computer science teaching practice which encourages flexibility of thought and is not only oriented to developing specific competences of a particular computer science environment.

For each one of these problems, we note that the teachers were able to find a solution by relying upon their knowledge of mathematics. Furthermore, the teachers also had the necessary computer science background to determine the solution to the problem in the computer-based environment available each time, because introduced to them on prior occasions, by searching for solutions to appropriately chosen problems which motivated and justified the use of this knowledge.

During the problem solving process within each environment, we asked the teachers to carefully perform an introspective analysis of the difficulties and hurdles encountered in the course of the activities and to clearly describe them, in order to subsequently carry out an examination and productive comparison of the different thought processes involved in the problem solving situation encountered.

Our research is founded upon the protocols written by the teachers and some interviews about them.

2. Analysis of the teachers' behaviours emerging in the problem solving situations.

Although not explicitly required, from the protocols produced we have remarked that almost all the teachers had initially developed solution strategies using pen and paper, relying upon algebraic language to describe the relationship between the data and the problem variables.

We note that an executive procedure was always connected to the various algebraic models for the determination of the results with pen and paper, based on operations whereby values were substituted and ordered calculations performed. This executive procedure is based upon the semantic command of the algebraic model by the person concerned, with regard to the operations involving the substitution of values, and to the command of the algebraic language syntax rules, for fast computation of the results.

Only at a later stage did the teachers attempt to construct a program according to the computer-based environment available at that moment in time.

To construct a Basic language program, the teachers made an attempt to consider the "point of view" of the executor-computer, coherently in accordance with the "conceptions that they had come of the operation of the device underlying the software used" which, from this point on, for brevity, we will refer to as the "conceptions of the device".

From the protocols we have noted that depending upon the level of development of one's own "concepts of the device" many different programs emerged which we classified under three types.

A first type was developed by those who had "mechanically" converted the algebraic solution strategy into a program, without attempting to restructure it.

For instance, in a problem of profit optimization for a calf breeding company, almost all the teachers formed an algebraic computational model as the following:

p=pi+a*i with i=30, i=60, i=90
p=pi+a*90+b*i with i=30, i=60, i=90
p=pi+(a+b)*90+c*i with i=30, i=60, i=90
p=pi+(a+b+c)*90+d*i with i=30, i=60, i=90

to describe the variation of the weight of the calves during the various months of the year on the basis of the initial weight "pi" of the calves and of the average daily increase of the weight "a", "b", "c", "d" in the four quarters.

Some of them translated this model into the Basic language by means of 12 assignment statements. These teachers demonstrated to have a very elementary "conception of the device", mainly based upon the effects of the execution of the individual statements of the language used. Their conceptions were therefore not very well structured, as they concerned limited aspects of device operation, and prevented the subject to put into practice cognitive processes able to systematically organise operative links between different statements.

Hence in the construction of the program, the executive model connected to the algebraic solution was imposed and the "conceptions of the device" developed by the subject constituted the cognitive aid for translating this solution into the available programming language.

A second kind was accomplished by those teachers who were able to also perform a more systematic thoughtful examination of their own "conceptions of the device". This enabled them to think about the algebraic solution produced, thereby finding different methods to express it in a program and to choose the most efficient one for computer execution. These teachers, for example, constructed four iterative structures to solve the problem outlined earlier.

The construction of the program still comes down to translating the algebraic model, because they use the iterative structures to substitute the time values into the various equations and to hence repeat the calculations. In this case, however, we observe that the teachers' "conceptions of the device" does not only constitute the aid on which the mechanical translation of the algebraic strategy relies upon, but constitutes a subject's operative thinking instrument, thus allowing her or him to acknowledge that something "is to be repeated" and to express "how" through the organisation and the connection of four loops. In this manner, these teachers demonstrate to have a more advanced "conception of the device".

The third kind of program was developed only by a limited number of teachers who are able to restructure the algebraic solution, transforming it, on the basis of their "conception of the device", into an efficient computer strategy.

They constructed a program in which the variation of the weight of the calves over the months is no longer seen as a function of time, expressed in days as in the algebraic model, but as a function of program execution, i.e. as a succession of the values expressed by an invariant loop.

We remark that the capability of the teachers to correctly handle an accumulator variable within a loop plays a significant role in the restructuring of the strategy. These teachers demonstrate to have an even more developed and articulated "conception of the device" which enables them to construct a solution strategy, by relying upon the new available language and to systematically organise operative links between the various statements.

It should be noted again that all the teachers had had numerous occasions, in previous training experiences and in classwork with pupils, to use iterative structures and accumulator variables for the construction and/or interpretation of simple programs and, therefore, that they apparently possessed the same level of computer-science knowledge. Their experiences were nonetheless not automatically translated into the capability to use such structures in a problem of greater difficulty.

We believe, that the different behaviours observed in the teachers are to be related to the various stages of development of their own "conceptions of the device" which appear to evolve through cognitive processes which lead to the construction of mental programming schemes.

From the observations of the behaviour of the teachers we had attempted to understand which were the cognitive processes able to develop their "conceptions of the device", so as to be able to construct mental programming schemes suitable for subsequent use in the construction of other more complex programs [7].

In the construction process of a programming scheme, we believe that the subject should:

- construct a space-time mental environment in which to organise, step-by-step, the simple actions that the executor must perform to determine the result of the problem. During this phase, the subject constructs her or his solution strategy and expresses it through a sequence of instructions, on the basis of the development of one's "conception of the device", very often "identifying oneself" in the executive role performed by the machine, in order to have a logical comparison of the correctness of one's own strategy.
- attribute a "single significance", in relation to a particular device available, to the sequence of instructions written to solve the specific problem encountered, acknowledging each instruction not only a specific role for the result that it singly produces, but also a functional role for the overall problem that it contributes to solve, within the entire sequence of instructions.

The construction process of a programming scheme changes and develops the subject's "conceptions of the device" only if during its construction a continuous dialectic is activated between two mental attitudes, with respect to the scheme and the parts thereof to be constructed: one within the process, to construct the sequence of instructions, and an external one to keep the existing relations between these instructions under control.

The absence of this dialectic between these two mental attitudes may cause the subject to produce a sequence of instructions to solve a specific problem, but not to give an overall meaning. In this way she or he is unable to make own the overall sense of the sequence produced, to develop one's own "conceptions of the device" and to use procedures as a programming scheme to appropriately recall in other contexts.

As generally occurs for all schemes, recalling a programming scheme is achieved every time a problem requiring a solution evokes, in someway, problems already dealt with which are similar with that to be solved and for which the subject recognises an efficient and pertinent scheme for the problem in question.

Recalling a programming scheme generally presents specific difficulties in the computer science environment as a result of the need to apply a standard structure to a problem context which presents proper characteristics. The major obstacles that the subject must overcome are due to the need, both to correctly interface the variables of the scheme with those of the problem to be solved and to integrate the structure of the scheme in the specific elaborations required by the problem.

In our experience we have had the possibility to confirm the assumption that many programming schemes are constructed functionally to suit the particular environment in which one operates and, as a result, they depend upon the language in which they are developed [4]

We have also observed that schemes may represent a form of "unflexible thought" when one passes to a new environment.

For instance, when we asked the teachers to develop and build a Pascal program for the problem of the calves, mentioned earlier, we noted that almost all of them translated old Basic program into Pascal syntax, without attempting to restructure it. We believe that this mainly occurred because the teachers, still not having developed an adequate conception of the device underlying to the new programming language, recalled programming schemes which they had memorised in the Basic language and were attempting to transfer them mechanically into the new programming environment.

In this manner, they appeared to have difficulty to abandon schemes, procedures and strategies of the Basic environment, in order to construct new and more functional ones to suit the new environment.

These teachers did not seem able to recognise when the resort to an already available scheme or procedure was functional to the solution of the problem and, therefore, its use justified by the need for "economy of thinking" and when, on the other hand, they had to abandon these mental schemes to put into practice new productive processes for the construction of a new and efficient solution strategy.

The schemes possessed, in fact, constituted a security for the subject and this was cause for additional problems for the teachers of the course, regarding the need to encourage the abandonment of unsuitable mental schemes for the new programming environment available.

The obstacle which emerged gains significant importance on educational grounds. It gives rise to the problem of how to balance the needs to encourage the construction of programming schemes within a certain language, without having the risk of inducing the subjects into the stereotype use of them when operating in a new environment.

When the teachers had to look for the solutions, in a Macintosh "Excel" spreadsheet environment, to the same problems proposed, they should have restructured the solution strategies formulated in the previous environments, adjusting them to meet the characteristics of this latter one, which required the space organisation of data and of the relationships to obtain the results in tabular form.

In this environment the variable acquires a space related meaning because it is associated to a cell [2], in turn identified by specific co-ordinates. A formula and the result of its calculation can simultaneously be contained within a cell, even if the program generally displays the result, by default.

We have noted that the beginner, working in this environment, immediately constructed her or his own conception of the characteristics of the software in use which, very often was not necessarily suited to the proper use of the instrument.

For example, the type of representation offered by a spreadsheet forced the teachers to interp the co-ordinates of a cell as the co-ordinates of the value contained within. Consequentl several teachers decided to insert or not the reference to a cell into a formula, on the basis of th actual value it contained and not on the basis of the meaning of the variable, associated to thi cell, within the formula that they are designing.

These teachers were "misled" by the fact that they saw only numerical values shown on th screen and consequently they employed their own solution strategy on the idea that the availabl software was alike a powerful calculator which offered the possibility to store all the calculatio sequences useful to attain the result. This forced them to recall arithmetic models with whic they attempted to find a strategy which suited their "conception of the device" and this led them in some cases, to not find the correct solution.

3. Some educational and research implications

From the foregoing, the behaviour of the teachers in problem solving situations with th computer seem to depend upon the "conceptions of the device" that they have developed an they appear to follow the development as the conception of the teachers evolve.

The training course has enabled the teachers to become aware of this fact, by comparing thei self-analyses and the examination of the cognitive processes involved. This has enabled them t recognise in their experiences a general characteristic, common to all participants, and to verif that several attitudes of the pupils could have been embraced by this view point.

The observations that we have made have provided us with useful elements to enable us to pus forward in our research work in the analysis of the characteristics that a teaching situation mus have to encourage the development of the pupils' ability in solving problems with the computer. In this context we believe that the need to take account of the "conceptions of the device", tha the pupils have developed, and of planning the teaching strategies, which encourage th development of these conceptions, is of crucial importance. From the results acquired durin the course, we are currently verifying whether even an incorrect "conception of the device" may be utilised to encourage the learning process in pupils.

We have, in this sense, designed, for instance, some teaching situations in a spreadshee environment in order to emphasise the rupture that occurs between an arithmetic and algebraic type approach in the solution to a problem. We are verifying whether it is possible to encourag an awareness, through them, of the differences between the two kinds of approach and to contribute to the construction of the concept of variable, evolving the conceptions of the pupil at the same time.

The observations made during the course, finally, offer elements to initiate research aimed at analysing the existing relationships between possible conditionings by the teachers and the development of pupil learning processes, in their classwork in problem solving situations with the computer.

Each teacher has her or his own "conception of the device" which conditions she or he in the work and which does not necessarily coincide with that which the pupils have developed. What the pupil learns is affected by the ideas of the teacher and by the choices which are made, in addition to the "computer culture" which the pupil makes own through her or his out-of-school experiences and which is, very often, different from that of the teacher.

In general, the "conceptions of the device" which each one develops are affected by attitudes and cognitive schemes that are constructed by a wide range of stimuli and cultural aspects.

It appears interesting for us to perform investigations in this sense, starting from those concerning the possible effects of the teacher's "conceptions of the device" on the pupil's existing "conceptions of the device". The largest difficulty, as we see it, concerns the instruments used for the analysis and the evaluation of the changes that have taken place in the pupils.

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ANALYSIS OF THE ACCOMPANYING DISCOURSE OF MATHEMATICS TEACHERS IN THE CLASSROOM

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SUMMARY

Our study examines what different teachers say, in class, during different activities, by analysing everything which does not deal exclusively with "mathematics". This leads us to check and characterise the variability of this discourse (called accompanying discourse), in particular by analysing its presumed functions.This allows us to have access (partial but in situation) to the "active" performances of the teachers.

We shall only present here the early stages of this research which is being carried out at the moment.

Methodology

We have worked out a methodology which allows us to characterize different aspects of this accompanying discourse and in particular to compare it from one activity to another and from one teacher to another.

We work on transcriptions of recordings made of teachers in class ; we have limited ourselves to one level ("seconde" classes, the first year of French lycées in which pupils have not yet chosen a scientific orientation) and to one chapter of the course of this class (homothetics).We insisted on recording all sessions spent on this chapter for each teacher invoved, in order to make any comparisons valid.

We have tried to collect these observations from quite varied classes, but we do not claim to have been exhaustive as this was not necessary considering our aims.

It is of course impossible to separate what the teacher says about mathematics or about what is happening in the classroom, from the actual contents of what is at stake, and from the organization, at least during the session under consideration, of the pupils' activities around these contents.

This is why we first characterize the proceedings of the whole of each session studied.

* linear description :

In this part we characterize all the activities which take place in the class, and cut them up into units which allow us to analyse the discourse of the teacher.

Each unit corresponds to a single well defined pupil-activity (finding the answer to an exercise (in class, in half-class groups, in small groups, in a test...), correction of a question in an exercise, of a problem, of a test, listening to a demonstration of the lesson, of a report or of a synthesis...)

This leads us to describe the session in groups of units, the overall reading of which allows us to have a fairly accurate idea of what happened concerning the mathematics contents.

This description in units (grouped together in the case where they concern successive solutions to the same exercise for example) is what we call the "linear description".

This cutting-up into units is based on the idea that it is more interesting to compare teachers' discourse which treats not only the same subject but also analogous activities within a comparable context. The analyses of the discourse are then carried out by unit.

* study by category

In this stage we pick out the succession of functions which we, the reader, attribute to the discourse which is given back to us, and then to study their distribution.

We label the phrases whose contents are not strictly mathematic keeping in mind their supposed function(s) in the teacher's discourse. If it seems possible to attribute several functions they are all indicated.

Here are the three types of functions we have retained, which do not exclude each other, although they are quite separate :

communication with the class, on a strictly non-mathematical level,

different titles or labellings (without justification),

commentary on the mathematical contents (at different levels, including on evaluation).

The parts of the discourse in which only mathematics are explained, without any commentary, are not labelled (which allows them to be distinguished in any case !).

These divisions are then measured in a vey elementary fashion : as we dispose of transcriptions uniformly typed for all teachers, we measured the length of the phrases in which appear a certain function and related them to the total length of the discourse (during the unit studied).

The relative frequency of purely mathematical discourse can thus be compared to others.

* longitudinal study

Here, on the contrary, throughout the discourse, a certain number of possible invariants of the discourse are picked out, such as the treatment by the teacher of memory (class or individual memeory), of analogy, errors, the use of questioning, repetitions, orders ....

On one hand a certain number of key words are picked out by units (you remember, as, false, ...), and on the other hand questions, repetitions, etc.... are studied systematically.

Following this, possible regularities are studied, for each teacher, by unit.

Intersection of these three studies allows us, firstly for each teacher, to make inferences about the representations and the contract, and then to compare these results with those for other teachers, both globally and locally for "comparable" units.

Initial results

1)The analysis by category of the discourse of three teachers in a "seconde" class on the chapter on homothetics (1984 curriculum) allowed the following results to be checked.

* the classification proposed is operational, even though several functions can be attributed to certain phrases (which was then done).

* for a single teacher

i) The relative distribution of the sentences concerning the different functions studied (communication, labelling, commentaries) varies according to the activities taking place (see table 1).

Table 1-1 : teacher E1, correction of construction exercises and correction of demonstration (in %)

	math	other	math	communication	labelling	commentary
E1 2.	54	46	54	28	10	8
E1 4.	73	28	73	15	4	9

Table 1-2 : teacher E3, class, collective search for demonstration, correction of demonstration (in %)

	math	other	math	communication	labelling	commentary
E3 4	52	48	52	17	20	10
E3 2	54	46	54	10	33	3
E3 1	52	48	52	8	27	13

Certain variations are due to the activity : it is not surprising that during a class (E3 4) there was more need to communicate with the class in general than during the correction of exercises (E3 1). Other , less obvious variations, must therefore be due more to the teacher !

ii) However, for a single "unit", this distribution is relatively stable for a single teacher, not only during a single lesson but also from one day to another.

More precisely, the distribution of mathematical discourse and the rest is stable for a single unit, but it is within discourse on mathematics that the proportions between more justified communication, labellings or commentaries may vary. (see table 2).

Table 2-1 : teacher E1, correction of construction exercises (in %)

	math	other	math	communication	labelling	commentary
E1 1.	63	38	63	20	12	6
2.	44	46	54	28	10	8
3.	61	39	61	11	20	8

Table 2-2 : teacher E1, correction of demonstration (in %)

	math	other	math	communication	labelling	commentary
E1 4.	73	28	73	15	4	9
E1 5.	69	32	69	8	16	8

Thus for teacher E1, the proportion of simply mathematical phrases remains stable (from 54% to 63%) for a single unit : ""correction of building exercises" (place in relation to lesson), and in relation to the detail of the non-mathematical discourse, the communication is either greater or not than the rest, whereas the share of commentary sentences is constant.

For another unit, "correction of demonstration", the proportion of mathepmatical phrases remains stable, (from 69 to 73%), and once again communication and labelling exchange places, whereas the share of commentary remains remarkably stable (the same as for another unit, and this is the only function which is repeated for which this is true.)

• For different teachers, for the same "unit" of activities, much greater differences exist than those picked out for a single teacher. In particular, an inversion may exist between the proportions of discourse dealing only with communication with the class and discourse dealing with mathematics. (see table 3).

Table 3-1 Teachers E1 and E2, correction of construction exercises (in%)

	math	other	math	communication	labelling	commentary
E1 1.	63	38	63	20	12	6
2.	54	46	54	28	10	8
E2 1.	70	29	70	6	3	20

Table 3-2 Teachers E1, E2 and E3, correction of demonstration (in %)

	math	other	math	communication	labelling	commentary
E1 4.	73	28	73	15	4	9
E1 5.	69	32	69	8	16	8
E2 2.	59	41	59	8	18	15
E3 1.	52	48	52	8	27	13

It can be seen that E1, for the two units considered, makes less comments than the others and tends more to communicate in a general way with the class. On the other hand, labelling functions are more frequent with E3, whereas the two functions are equal with E2 or tend more towards commentary.

Highly contrasted types of discourse can thus be noted.

In conclusion, we demonstrate that each teacher has his own style, but what varies is not a priori neutral : the differences play in particular on the relative length of the commentaries on mathematics in relation to their simple exposition and to the relative length of general communication with the class.

Very different accompanying discourse has been noted : on one hand, that including mostly commentary, on the other hand, that in which communication with the class predominates.

The stability observed with a single teacher allows us to consider the problem of access to underlying performances which may contribute to class practice.

For example we may wonder whether those teachers who favour communication consider that learning happens first through an active attitude, or a good relationship with the teacher, whereas the others feel that they can base themselves more on a reflexion on mathematics.

For this reason it will be interesting to complete this first level of analysis by a closer study of teacher discourse.

2) Longitudinal analysis (teacher E2)

* regular features of the forms used

We have noted the existence of a certain relationship between the contents of the teacher's discourse and the forms he uses : he uses more impersonal forms when talking about an aim which is distant (*you will have to, it must, you are asked to*...), and uses orders more when immediate actions are required (*take..., choose..., draw*...) These regular features are undoubtedly not specific to a mathematics class, but can, however, function as markers.

On the other hand "I" arises in order to describe the teacher's action (*I draw, I construct*...) or to signal regain of control over the class, especially concerning an institutionalisation (*I have shown that*...).

The present tense (active) is used especially to call on memory (*you remember that , you know that*...).

* questions

They are often a bit vague : *"what do you need ?", "how do you show that ?"*. Type of answer (given by the teacher) : *"because we have vectors, we have to use them"*.

The response is a type of recipe out of context, more than a precise method, in as much as such methods are not presented explicitly in the lesson and that therefore only recipes, out of context, are (possibly) available. One can understand why pupils sometimes do not reply...

* doubt

Not frequently present, the only occurences being for example *"are you sure that .... ?"* which has such negative connotations that it suggests the answer and not doubt.

* repetitions

They seem to be there to force the pupils to listen, to re-direct ther attention. These are exact repetitions, a sort of "encore" ...

"You remember, don't you remember ...", "F transforms , it transforms.....".

* summoning memory

This is generally a time marker, the reference chosen being not the contents but the moment when what is being sought was first encountered :

"you saw in "troisième" (preceding year) , in the beginning of the year we saw..."

Certain labellings are more closely linked to the contents, such as *"in the lesson on vector theory"*, but this can also be read as a reference to the progress of the class. This is perhaps another example of the fact that the teacher has not constructed other reference points with the pupils, which might be more closely linked to the contents (methods, frames, tools...). He only has at his disposal anecdotal, or factual memories, linked to actions. Let us quote, for example :

"you remember, we saw it all right at the beginning of the year, ..., I even told you to remember this formula, as its often used, it's very easy, you even had it in your test..."

The "present active" mode implicitly returns the task to the pupils : the remembering is on their side, their responsibility. Moreover it seems to contribute to the appropriation of knowledge, and as it can only be conveyed by discourse one can perceive in it one of the important functions of his discourse for this teacher.

- analogy

Analogy is frequently used, but it is not formalised as such (which is logical given what precedes this). "Doing the same thing" occurs on the level of the action : *"we've already done this, the demonstration is the same", "how did we do it ? It was always the same", "I have shown that... show in the same way that".*

One can note the frequent ocurrences of the verb "to do", a key word in this teacher's discourse...

- the implicit

A certain number of examples can be noted, which are more or less transparent, and linked to the teacher's judgements. For example *"do these drawings quickly"* could indicate that the teacher does not consider this form of activity to be very important (easy = quick).

The importance of the labellings already picked out in the study by category on one type of exercise (labelling in time, as we can now point out), and of commentary (which we now know to be markers linked to action, to the progress of the class, rather than real method).

We can now therefore deduce from our analyses a teacher's habits, expressed in his discourse. Can these habits have any consequences on the learning (and if so, which)? These are questions which further research will study.

Conclusion

As this study is being carried out at the present time, we cannot claim to conclude too hastily. We can, however, already confirm that accompanying discourse varies in a non-anecdotal manner from one teacher to another: regular differences can be discerned regarding the presumed functions of this discourse (we find more or less general communication with the class, a greater or smaller number of markers given without commentary, more or less justified commentary of different types).

Thus the different ways of seeing the learning and teaching of mathematics seem not only to influence the overall choices of the teachers (organization of contents and activities) but also seem to show through in their everyday discourse and may influence these choices

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VAN HIELE LEVELS OF LEARNING GEOMETRY

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According to the theory of Pierre and Dina van Hiele, students progress through levels of thought in geometry. Both researchers and curriculum developers base their work on these levels. This presentation will review extant theoretical and empirical work on the van Hiele theories, organized around critical questions and issues.

According to the theory of Pierre and Dina van Hiele, students progress through levels of thought in geometry (van Hiele, 1986; van Hiele-Geldof, 1984). Thinking develops from a Gestalt-like visual level through increasingly sophisticated levels of description, analysis, abstraction, and proof. This paper will raise critical questions about this theory and review relevant theoretical and empirical work that addresses these questions.[1]

How accurate is the van Hiele level theory in describing geometric thinking?

Generally, empirical research has confirmed that the van Hiele levels are useful in describing students' geometric concept development, from elementary school to college (Burger & Shaughnessy, 1986; Fuys, Geddes, & Tischler, 1988; Hoffer, 1983; Wirszup, 1976). For example, Usiskin (1982) found that about 75% of secondary students fit the van Hiele model (the percentage classifiable at a level varied with the instrument and scoring scheme). Research supports the existence of unique linguistic structures at each level; for example, "rectangle" means different things to students at different levels (Burger & Shaughnessy, 1986; Fuys et al., 1988; Mayberry, 1983). Thus, levels appear to describe students' geometric development.

How discrete are the levels; do hypothesized discontinuities exist?

Russian research intimates discrete levels (Hoffer, 1983; Wirszup, 1976); however, results from multiple other sources are mixed. Several researchers have reported that students, especially those in transition, are difficult to classify reliably, especially for levels 2 and 3 (Burger & Shaughnessy, 1986; Fuys et al., 1988; Usiskin, 1982). Fuys et al. (1988) used instructional-assessment interviews of 6th-9th graders, which allowed them to chart students' ability to make progress within and between levels as a result of instruction. Whereas some of the students appeared to be on a plateau, there were also those who moved flexibly to different levels during the teaching episodes.

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[1] Both the number and numbering of the levels have been variable. We shall base this review on the van Hieles' original five levels and later discuss a sixth. Our rationale is twofold. First, although van Hiele's recent works have described three rather than the original five levels, both empirical evidence and the need for precision in psychologically-oriented models of learning argue for maintaining finer delineations. Second, the empirical evidence also suggests a level that is more basic than van Hiele's "visual" level. (There have been several different numbering systems used for the levels; we have adopted one system and have transposed those of each researcher to this scheme for consistency's sake.) Space constraints prohibit description of the levels here; see [Clements, in press #585; Fuys, 1988 #546; Hershkowitz, 1990 #626; van Hiele, 1986 #39; van Hiele-Geldof, 1984 #547].

Do students reason at the same van Hiele levels across topics?

This question is also relevant to the issue of the discreteness of the levels, and evidence on this question is similarly mixed. A test of consensus revealed that preservice elementary teachers are on different levels for different concepts (Mayberry, 1983), as are middle school (Mason, 1989) and secondary students (Denis, 1987). Similarly, Burger and Shaughnessy (1986) reported that students exhibited different preferred levels on different tasks and oscillated from one level to another on the same task under probing. The researchers thus characterized the levels as dynamic rather than static, and continuous rather than discrete. Gutiérrez and Jaime (1988) compared the level of reasoning of preservice teachers on three geometric topics: plane geometry, spatial geometry (polyhedra), and measurement. The levels reached across topics were not independent, but the data did not support the theorized global nature of the levels. The researchers hypothesized that as students develop, the degree of the globality of the levels is not constant, but increases with level. That is, as children develop, they grasp increasingly large "localities" of mathematical content and thus understand larger areas of mathematics. Fuys et al. (1988) agreed that when first studying a new concept, students frequently lapsed to level 1 thinking. They maintained, however, that students were quickly able to move to the higher level of thinking they had reached on prior concepts. The researchers therefore claim that these results support the contention that a student's *potential* level of thinking remains stable across concepts.

Do the levels form a hierarchy?

Research more consistently indicates that the levels are hierarchical, although here too there are exceptions (Mason, 1989). For example, Mayberry (1983) employed Gutman's scalogram analysis to show that her tasks representing the levels formed a hierarchy for preservice teachers. These results were replicated by Denis (1987) for Puerto Rican secondary students. Gutiérrez and Jaime (1988) reported similar analysis and results, but only for levels 1 to 4; level 5 was found to be different in nature from the other levels. Most other researchers did not test the hypothesis in a similarly analytic manner; nevertheless, they interpret their results as supporting this hypothesis (Burger & Shaughnessy, 1986; Fuys et al., 1988; Usiskin, 1982). Thus, the levels do appear to be hierarchical; however, this does not imply a maturational foundation. First, assignment to levels does not seem to be strictly related to age (Burger & Shaughnessy, 1986; Mayberry, 1983). Second, development through the hierarchy proceeds under the influence of a teaching/learning process.

What is the most basic level?

There is evidence for the existence of a level more basic than the van Hiele's "visual" level (Battista & Clements, 1987). For example, 9-34% of secondary students have failed to demonstrate thinking characteristic of even the visual level; 26% of the students who began the year at level 0 remained at level 0 at the end of the year (Usiskin, 1982). Such stability argues for the existence of level 0 (Senk, 1989). Likewise, 13% of the response patterns of preservice teachers do not meet the criterion for level 1 (Mayberry, 1983). Finally, students who enter a geometry course at level 1 perform significantly better at writing proofs than those who enter at level 0 (Senk, 1989). This issue is not resolved, however. Fuys et al. (1988) specified that to be "on a level" students had to consistently exhibit behaviors indicative of that level. They also state, however, that level 1 is different from the other levels, in that students may not be able to exhibit the corresponding behaviors (i.e., they may not be able to name shapes). However, most evidence from the van Hiele and Piagetian perspectives indicates the existence of thinking more primitive than, and probably prerequisite to, van Hiele level 1. Therefore, we postulate the existence of a "level 0." At this pre-recognition level, children perceive geometric shapes, but perhaps because of a deficiency in perceptual activity, may attend to only a subset of a shape's visual characteristics. They are unable to identify many common shapes. They may distinguish between figures that are curvilinear and those that are rectilinear but not among figures in the same class. That is, they may differentiate between a square and a circle, but not between a square and a triangle. Students at this level may be unable to identify common shapes because they lack the ability to form requisite visual images. These images presuppose mental representations constructed from the child's own actions.

Should other characteristics of the levels be considered?

Levels are complex structures involving the development of both concepts and reasoning processes. In addition, researchers have emphasized the importance of several interrelated notions: intent, belief systems, and metacognition. Students must become aware of what is expected, intentionally thinking in a certain way. For example, students on higher levels used such language as "explain," "provide it," and "be technical" in justifying their reasoning. Students on lower levels believed that they should respond to a task on paper exactly as it appeared (e.g., changing its orientation is not allowed). More of these students labeled an obtuse triangle a "triangle" when using a physical manipulative triangle (Fuys et al., 1988).

What levels of thinking are evinced under "traditional" instructional paradigms?

Studies of students by Pyshkalo and Stolyar indicated a significant number of Russian students were perceiving shapes only as wholes. Students stayed at level 1 for a considerable time; by the end of grade 5, only 10-15% reached level 2 (Russian students enter grade 1 at 7 years of age). This delay was even greater with respect to solids, for which there was no noticeable leap until the 7th grade (Pyshkalo, 1968; Wirszup, 1976). Fuys (1988) found that 19% of U.S. 6th graders were uniformly at level 1 even after instruction. They gained but a little level 1 knowledge, visual thinking about shapes and parallelism, from work with manipulatives. Another 31% made progress within level 1 and were progressing toward level 2. The final 50% began with level 2 thinking and progressed toward level 3. Some made deductive arguments, but most equated "proof" with generalization by examples. Secondary students in the U.S. who have studied geometry formally are nonetheless on levels 0 to 2, not level 3 or 4; almost 40% end the year of high school geometry below level 2 (Burger & Shaughnessy, 1986; Suydam, 1985; Usiskin, 1982). These students may not benefit from work in formal geometry if their knowledge and that presented to them are organized differently.

What levels of thinking do traditional textbooks reflect?

Fuys et al. (1988) analyzed several current geometry curricula as evidenced by U.S. text series (grades K-8) in view of the van Hiele model. Most work involved naming shapes and relations like parallelism at level 1. Students were only infrequently asked to reason with figures. Students would not need to think above level 1 for almost all of their geometry

experiences through grade 8. Topics were repeated across grades at the same level, and properties and relationships among polygons were sometimes taught incorrectly. Similar analyses of older Russian textbooks (those written before several major reforms) revealed the absence of any systematic choice of geometric material, large gaps in its study, and a markedly late and one-sided acquaintance with many of the most important geometric concepts (Wirszup, 1976). Only about 1% of all problems dealt with geometry. This left grade 6 students doing work corresponding to the first three levels of geometric development simultaneously.

Critical issues

There are problems with the research on the veracity of the theorized levels. For example, Fuys et al. (1988) interpret their results as supporting both the 5-level and the new 3-level model—visual (previously level 1), analytic (previously 2), and theoretical (previously 3-5). They caution that the three-level model may not be sufficiently refined to characterize thinking (e.g., students progressed toward level 3 but with no sign of axiomatic thinking). Further, van Hiele appears to describe the new visual level as combining aspects of the previous levels 1 and 2; therefore, the mapping from one model to the other is not unambiguous. Furthermore, if levels can be changed and combined, their hypothesized discrete, hierarchical psychological nature must be questioned. The question is, how wide a band can be permitted before the notion of hierarchical dependency disintegrates?

Further, should students' thinking be characterized as "at" a single level? For example, Gutiérrez, Jaime, and Fortuny (in press) attempt to take into account students' capacity to use each one of the four van Hiele levels, rather than assign a single level. They use a vector with four components to represent the degree of acquisition of van Hiele levels 1 through 4 (e.g., one student might have a grade component for level 1 of 96.67%; level 2, 82.50%; level 3, 50.00%; and level 4, 3.75%). They found many students who are apparently developing two consecutive levels of reasoning simultaneously, and hypothesized this results from mathematics instruction that leads students to begin the acquisition of level $n + 1$ before level n had been completely acquired (Gutiérrez et al., in press). Such alternate conceptualizations of levels of thinking need to be explored; they too may bring into question the nature of the levels.

The way in which students, especially young students, learn geometric concepts has also been questioned. First, research demonstrates that young children can discriminate some of the characteristics of shapes, and often think of two-dimensional figures in terms of paths and motions used to construct them (Battista & Clements, 1987; Clements & Battista, 1989; Clements & Battista, 1990; Kay, 1987). This is inconsistent with the levels as presently conceived. Second, while young children are currently taught with a "template" (visual prototype) approach to recognizing geometric patterns, Kay (1987) maintains that this is appropriate only if there is basically only one such template for each class (e.g., squares)—but that this does not apply to hierarchical-based classes. In contrast, Kay provided first graders with instruction that began with the more general case, quadrilaterals, proceeded to rectangles, and then to squares. It addressed the relevant characteristics of each class and the hierarchical relationships among classes and used terms embodying these relationships ("rectangle-quadrilateral"). At the end of instruction, most students identified characteristics of quadrilaterals, rectangles, and squares, and about half identified hierarchical relationships among these classes, though none had done so previously. Kay states that concepts are initially understood through inductive processes if the definition of the concept involves a complex deductive argument, but visually the concept can be represented by a small number of templates (e.g., "circle"). If the definition of the concept involves a relatively simple deductive argument and the concept cannot be represented easily by a template, then initial understanding will be deductive (e.g., "quadrilateral"). This dichotomy is similar to Vygotsky's (1934/1986) formulation of spontaneous versus scientific concepts. While both the depth of Kay's first graders' understanding and the generalizations made on the basis of the empirical results must be questioned, such alternate hypotheses deserve further investigation. Future investigations should ensure that students are not simply mirroring repetitious verbal training; "Direct teaching of concepts is impossible and fruitless. A teacher who tries to do this usually accomplishes nothing but empty verbalism, a parrotlike repetition of words by the child, simulating a knowledge of the corresponding concepts but actually covering up a vacuum" (Vygotsky, 1934/1986, p. 150).

In a similar vein, De Villiers (1987) concluded that, in contradiction to van Hiele's theory, hierarchical class inclusion and deductive thinking develop independently, and that hierarchical

thinking depends more on teaching strategy than on van Hiele level. He then describes a successful teaching strategy in which eighth and ninth grade students were taught first about quadrilaterals, and how special quadrilaterals could be obtained by specifying properties. This approach was contrasted with the traditional approach in which students associate names of figures with visual prototypes. "The defining quality associated with the name is therefore determined by the visual perception of the figure.... We believe that the observation that children think of shapes as a whole without explicit reference to their components, is the direct result of our actually teaching children from the start to think of shapes as a whole and in terms of visual prototypes, and with no reference to their components" (p. 19). However, the author makes this claim based on experiments with students intellectually capable of attending to properties. Our research, for instance, indicated that after being taught about the properties of squares and rectangles, if asked why they say that squares are special kinds of rectangles, many first graders say simply "because the teacher told us." However, the criticism of the van Hiele levels raised, that the levels are very dependent on the curriculum, is certainly worthy of further research.

Such questions and issues lead to the conclusion that van Hiele research has added to our knowledge considerably; however, in the future the corpus might be structured to test simultaneously alternate hypotheses (for example, finding students whose behavior seems to support a characteristic of the levels does not provide a strong test of competing hypotheses for the given behavior).

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SOME THOUGHTS ABOUT INDIVIDUAL LEARNING, GROUP DEVELOPMENT, AND SOCIAL INTERACTION

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A brief analysis is made of a ten minute episode in which three eight year old children worked together to solve arithmetical tasks. The notion of a taken-as-shared basis for mathematical activity that makes communication possible is discussed, and a distinction is made between equivalent and parallel interpretations that each involve the experience of intersubjectivity. It is argued that individual learning and the development of a taken-as-shared basis for mathematical activity are reflexively related in that students learn as they participate in the interactive constitution of the situations in which they learn. The paper concludes with a discussion of the reflexive relationship between mathematical meaning and mathematical symbols.

The primary focus of this paper is on the relationship between individual students' learning as it occurs in social interaction and the development of the group's taken-as-shared or consensual basis for mathematical activity. To this end, an analysis will be presented of a short episode in which three second grade girls, Karen, Jan, and Marie, worked together to solve a sequence of arithmetical tasks presented diagrammatically in a balance format. Prior observations indicated that there were differences in their general orientations to mathematical activity. Karen and Jan appeared to consistently experience arithmetical computation as an activity in which they acted on numbers as arithmetical objects. In other words, they seemed to experience the signifieds of conventional arithmetical symbols as having a concrete, manipulable quality. As a consequence, an explanation or justification was for them an account of actions on arithmetical objects. In contrast, arithmetic was for Marie primarily an activity in which she followed procedural instructions for acting on arithmetical symbols which did necessarily symbolize anything beyond themselves. From her point of view, to explain or justify was to specify the procedural steps of an instruction. There was therefore an incommensurability in the children's general conceptions of or beliefs about arithmetical activity.

Equivalent and Parallel Interpretations

The first task presented on the activity sheet involved specifying the number that balanced three twenties and was relatively routine for the three children. Jan counted "20, 40, 60", and the other two children accepted her solution without discussion. This unanimity is itself revealing, suggesting as it does that each child assumed without question that her interpretation of the task as an additive situation was shared by the other two. Further, we, as observers of the episode, have no evidence that would allow us to discriminate between their individual interpretations. Equivalent interpretations of this type can be differentiated from parallel interpretations. In the latter case, each participant of an interaction assumes that his or her interpretation is shared by the others but we, as observers, have evidence suggesting that there are differences in their individual interpretations. From the participants' points of view, they each experience intersubjectivity with the others when they make both equivalent and parallel interpretations and, consequently, their interpretations are said to be consensual or taken-as-shared in both cases. The difference between the two depends on whether we, as observers, can infer that there are implicit incommensurabilities in their interpretations. On occasion, the participants can become aware of discrepancies in their individual interpretations, at which point an issue that was taken-as-shared becomes an explicit topic of conversation (Billig, 1987), thus giving rise to possible learning opportunities. This is what happened when the children attempted to solve the second task which involved balancing three nineteens.

Karen: What's nine and nine?
Marie: Nine and nine is...eighteen...and nine more...
Jan : And...
Marie: (Counts on her fingers) Nineteen, 20, 21, 22, 23, 24, 25, 26, 27. Twenty seven.

Karen's initial question indicates that she partitioned each of the three nineteens experienced as arithmetical objects into the numbers ten and nine. As her subsequent solution will substantiate, she intended to find the sum of the three nines and add it to thirty, the sum of the three tens. Marie seemed to understand Karen's intent, for she not only answered her question but anticipated the next step in her solution. Marie and Karen had therefore made taken-as-shared interpretations and were engaged in joint mathematical activity at this point in the episode. Marie completed her solution by counting the "1"s of the three 19's, "Twenty seven, 26...27, 28, 29, 30." This indicates that she had partitioned the two-digit numeral "19" per se rather than the number 19 as an arithmetical object, the result being one and nine rather than ten and nine. As observers, we can therefore infer retrospectively that Marie's and Karen's interpretations of the task were parallel rather than equivalent.

As the episode continued, Karen interpreted Marie's answer of thirty as a partial result, that of adding three tens:

Karen: Thirty...Okay, there's thirty there...now a ..now add all the nines to thirty.
Marie: No!
Karen: Mmm hmm [Yes]...'cause there are thirty (points to the activity sheet). If you take all the nines off, those are thirty.
Marie: But wait, look...
Karen: That's easier.

Karen's direction to "add all the nines to thirty" made no sense to Marie because she had already added them when she arrived at her answer of 30. At this point, incompatibilities in their previously taken-as-shared task interpretations became apparent and they were in a situation of interpsychological conflict. In general, such conflicts arise against a background of consensual understandings (Wittgenstein, 1964). Thus, Karen and Marie could still communicate and attempt to resolve their differences because the goal of adding three nineteens was consensual, as was the act of adding the three nines to arrive at 27. Attempts to resolve such conflicts can be interpreted in terms of both learning opportunities for individual children and as opportunities for the group to elaborate its taken-as-shared understandings (Maturana, 1980).

In this particular case, Karen interpreted Marie's rejection of her solution as a challenge and attempted to resolve the interpsychological conflict by justifying her solution: "If you take all the nines off, those are thirty." In doing so, she described the result of performing an action on numbers experienced as arithmetical objects. This justification made no sense to Marie because, for her, the result of taking all the nines off was three. However, rather than questioning Karen's justification, she instead attempted to justify her own answer of thirty. To this end, she first wrote three 19's in traditional textbook column format, saying "One, one, and one. Nine, nine, and nine...plus, and equals...it's thirty." Her last comment, "It's thirty," strongly indicates that she expected the answer would be thirty and was going to calculate to demonstrate this to Karen. In other words, calculation as an act of following procedural instructions was a means of justification for her in this social situation. She was therefore surprised and experienced intrapsychological conflict when she attempted to calculate by using the standard algorithm and found that her conviction in her answer was unfounded. This conflict can be seen to have a social genesis in that Marie would not have attempted to justify her answer but for the interpsychological conflict with Karen (Krummheuer, personal communication). This observation does not however mean that the social aspect of mathematical learning is limited solely to that of face-to-face interactions serving as possible catalysts for students' individual, autonomous cognitive constructions. The intrapsychological conflicts that students experience arise against the background of their current mathematical understandings, themselves constructed in the course of their prior mathematical acculturation. Consequently, both the personal mathematical problems that students attempt to resolve and the solutions they construct are constrained by their participation with others in the constitution of a taken-as-shared basis for mathematical activity (Solomon, 1989). This is the case even when students work alone and do not interact with each other or the teacher for a period of time (Scribner, 1990).

Returning to the episode, it is apparent that Karen remained convinced that she knew how to solve the task and demonstrated her solution method: "It's ten, twenty, thirty ... (she then counts on her fingers) 31, 32, 33, ... 55, 56, 57." Marie rejected Karen's explanation and began to count by ones on a hundreds board. At this point, the teacher joined the children and suggested that the task could be related to the previously solved task of adding three twenties. Marie, who had been an observer of the ongoing exchange between Karen and Marie, then solved the task, "Twenty, twenty, twenty would be sixty ... take away three from sixty and that's your answer."

Marie, however, thought the answer would be 59, presumably reasoning that if each addend decreased by one then the sum would decrease by one as well. This was consistent with her answer to the next task, which involved adding three twenty ones. Jan and Karen solved this task by making equivalent interpretations as they interactively constituted a joint solution in which they added three to sixty, the sum of three twenties. This was a novel solution for Karen and probably stammed from her observation of the interaction between Jan and the teacher. Marie then proposed sixty one as the answer and Karen challenged her, "Yeah but there's three ones. There's not one. There's three." Again, the differences in their conceptual possibilities and beliefs about the general nature of mathematical activity were such that they could not arrive at a consensus.

Marie initiated the discussion of the fourth task which involved adding four twenty ones by saying "Ten ... there's twenty, forty, sixty ..." This was a novel solution attempt for her and suggests that she might have been attempting to adapt her mathematical activity so that it would be acceptable to Karen. On hearing Marie's comments, Karen interpreted the task as that of adding four to eighty. It would therefore seem that Karen's task interpretation was influenced by Marie's attempt to accommodate to her (Karen's) mathematical activity. The two then engaged in joint activity in which they constructed parallel rather than equivalent interpretations. In particular, it appeared that Marie's adaptation was to name the twos that resulted when she partitioned the numeral "21" as "twenty, forty, sixty." Jan meanwhile constructed an interpretation equivalent to Karen's when she observed the exchange between Karen and Marie and solved the task by adding four to eighty. She then explained what was for her a novel solution and Marie had an insight.

> Marie: I know something, I know something. Okay, two and two, and two and two. Four, six, so ... sixty, hey look ... two and four. Two plus two is four, plus two more is six, and eight, and it's eighty four.

Her insight seemed to involve the construction of a relationship between two number word sequences (i.e., "six" - "sixty", "eight" - "eighty"). In adapting her mathematical activity in this way, Marie constructed a procedure for acting on number words and numerals that might make it possible for her to be effective as she interacted with Karen and Jan. Her observable solution was indistinguishable from theirs and, in fact, she seemed to believe that she understood how they had solved the task. However, on the basis of this analysis, it seems reasonable to conclude that her task interpretation was parallel to Karen's and Jan's equivalent interpretations.

Collective Development

Despite the difficulties stemming from the incommensurability of their beliefs and their differing conceptual possibilities, each child did construct personally novel solutions in the course of

the ten minute episode that they would not , in all probability, have produced had they worked alone. It is also possible to talk of the development of the group by focusing on the development of taken-as-shared mathematical interpretations. Recall, for example, that at the beginning of the episode Karen partitioned numbers when she interpreted tasks whereas Marie attempted to use her version of the standard algorithm. Jan for her part worked independently of the other two children. The consensual basis for their mathematical activity seemed to involve little more than an unstated agreement that these particular tasks were additive situations. By the end of the episode, their mathematical activity was taken-as-shared in that each experienced intersubjectivity with the other two. As has been noted, Karen's and Jan's task interpretations were equivalent whereas Marie's was parallel to their's. The non-linearity of interactional processes becomes apparent once it is observed that the development of this consensus cannot be attributed to the actions of any one child. Instead, it was a phenomenon that emerged in the course of their social interactions and, for this reason, we talk of the taken-as-shared basis for mathematical activity as being interactively constituted (Bauersfeld, 1988; Voigt, 1985).

Consistent with the view that communication is a process of mutual orientation rather than of the transmission of information, the children's taken-as-shared basis for mathematical activity can be seen to have evolved as they each made adaptations which eliminated discrepancies between their own and others' mathematical activity while pursuing their goals. In other words, the process of interactive constitution occurred as the children attempted to avoid miscommunicating by influencing each other's mathematical activity and while simultaneously being influenced by their interpretations of that activity (Bauersfeld, 1980). From this point of view, the interdependence between individual and group development is considered to be reflexive in nature. On the one hand, the children's individual mathematical activities were constrained by their participation in the interactive constitution of a taken-as-shared basis for mathematical activity. On the other hand, this evolving basis for mathematical activity did not exist apart from and was interactively constituted as each child attempted to coordinate her mathematical activity with that of the other two. In short, we might say that the children learned as they participated in the interactive constitution of the situations in which they learned.

Syntax and Semantics

In the course of the analysis, learning was inferred to have occurred only when the children's adaptations contributed to their acculturation into the mathematical practices of wider society. This point can be further clarified by considering the relationship between what is traditionally called syntax and semantics. Karen solved the third task which involved adding three twenty ones by relating it to the sum of adding three twenties. Presumably, her solution was based in part on a syntactic relationship, namely that "21" is the immediate successor of "20" in her forward number word sequence. However, it seems implausible to argue that she first constructed a purely syntactic relationship and then gave it semantic meaning. Her's was a mathematical reality in which she acted on experientially real mathematical objects, and in such a reality where numbers can be partitioned, the successor relationship *means* that the number 21 is 20 and one more. In other words, her task interpretation transcended the separation of syntax semantics (cf. Winograd & Flores, 1986). Given the relative sophistication of her conceptual operations, she immediately saw that 21 was one more and that consequently to find the sum "you just...you just add three" to sixty. If the distinction between syntax and semantics is maintained in such instances, we could say that they are reflexively related in that the syntactic relationship constrained her semantic interpretations, and her semantic understanding made the syntactic relationship significant as she attempted to solve the task (cf. Laborde, 1982).

It would also seem that syntax and semantics were also inextricably intertwined in Jan's case when she solved the last task by first adding four twenties and then adding four to the sum. Her conceptualization of the number twenty one as twenty and one was probably supported by a regularity she had abstracted from number word or numeral sequences. In particular, for her, partitioning the number word "twenty one" was synonymous with partitioning the number it signified. In contrast, partitioning a numeral or number word did not carry the significance of acting on an arithmetical object for Marie. Her learning can be accounted for by the abstraction of regularities between number words or numerals per se and consequently did not involve the integration of syntax and semantics. Learning of this type is of course not that aimed at by most mathematics educators and can be characterized as instrumental in Skemp's well known terminology.

Thus far the role of syntax and semantics in the children's learning has been discussed primarily in cognitive terms. If learning is viewed as a process of acculturation, we might say that Marie constructed notational regularities that are consensual with those constructed by acculturated members of society. Jan and Karen, in contrast, constructed symbolic mathematical practices consensual with those of wider society (cf. Bakhurst, 1988; Geertz, 1983). In the latter case, the term symbolic of course implies that syntax and semantics are interrelated reflexively. More generally, this conception of symbolic activity draws attention to a central aspect of mathematical experience discussed by Kaput (in press).

> One can draw an analogy between the way the architecture of a building organizes our experience, especially our physical experience, and the way the architecture of our mathematical notation systems organize our mathematical experience. As the physical architecture constrains and supports our actions in ways that we are often unaware of, so do mathematical notation systems. (p. 4)

In the viewpoint being advanced here, it is not mathematical notation per se that constrains our thought but rather our acculturation into consensual symbolic mathematical activities. To the exte that syntax and semantics are reflexively interrelated, the process of learning a mathematical notation system involves the construction of a experiential mathematical reality that is consensual with those of others who can be said to know mathematics. Consequently, when we use conventional notation individually or while communicating with others, we participate in the continual regeneration of a taken-as-shared mathematical reality, and that consensual reality constrains our individual ways of thinking and notating. Both Mehan and Wood (1975) and Brune (1986) have drawn attention to the more general phenomenon that we create the world about which we speak as we speak about it. The reflexivity of linguistic activity in general and of symbolic mathematical activity in particular indicates that Kaput's analogy is appropriate. The bas metaphor we (and Jan and Karen but not Marie) implicitly use when interpreting mathematical notation is that of acting in physical reality (Bloor, 1983), and we see the abstract metaphorical reality we create symbolically as we look through the notation we interpret.

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UNE ANALYSE DES BROUILLONS DE CALCUL D'ELEVES CONFRONTES A DES ITEMS DE DIVISIONS ECRITES

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ABSTRACT

This paper discusses research still in progress supported by the Swiss National Fund for Scientific Research, no 11-25448-88, entitled: L'étude des algorithmes de calculs dans la transmission et la constitution des connaissances numériques, , conducted by J. Brun, F. Conne, and R. Schubauer, of University of Geneva, and J. Retschitzki, of University of Fribourg. By this presentation, we want to present our research about the teaching of written division at primary school. In the following paper, we describe the frames of our research, and the main lines of our results. These are:

- A classification of errors observed in written division, which envolve errors of division, but also errors of subcalculations: multiplications, subtractions, and additions.

- A description of the links the pupils make, in their calculation, between division and multiplication.

- A description of the links between these errors and the teaching method of division.

- All these results are based on the analysis of the pupils draft of their calculations.

Our purpose for this PME session will be the illustration by examples and commentaries of these results.

L'objet de cette recherche concerne les relations qui existent entre les connaissances numériques des élèves et les algorithmes de calcul qui leur sont enseignés.

Par algorithme de calcul nous désignons " une suite finie de règles à appliquer dans un ordre déterminé à un nombre fini de données pour arriver avec certitude (c'est-à-dire sans indétermination ou ambiguité) en un nombre fini d'étapes, à un certain résultat et cela indépendamment des données. " (Conne 1988).

Par connaissances numériques de l'élève, nous signifions l'ensemble de ses connaissances conceptuelles et procédurales lui permettant de faire les mises en relations nécessaires à l'exécution de différentes tâches en situations d'enseignement ou non. L'élève se construit sur une longue durée ces connaissances, à l'école et hors de l'école; celles-ci peuvent être classées en dénombrements, comptages , calculs symboliques et calculs sur un diagramme, ... (Conne 1988).

Dans l'algorithme d'une opération écrite nous retenons deux ordres de significations :

- l'un , numérique, qui renvoie aux opérations et relations élémentaires
- l'autre, numéral, qui renvoie aux règles de l'écriture des nombres et à la disposition des calculs, ainsi qu'à la configuration des écritures des nombres.

L'opération écrite de division est le support de cette étude des relations entre connaissances numériques/numérales de l'élève et algorithmes de calcul . Les principales caractéristiques de cette opération de division sont les suivantes :

- la part "évaluative " du calcul
- la possibilité d'un reste ; tout n'est pas résolu avec les "tables"
- le sens (direction) dans lequel on traite le nombre écrit : de gauche à droite au lieu de droite à gauche pour les autres opérations déjà apprises. Les opérations intermédiaires de soustraction requièrent que l'on coordonne les deux sens.
- l'enchainement de sous-algorithmes : ceux de la multiplication et de la soustraction.
- la division "exacte" demande une ouverture vers le dépassement des nombres entiers et des quantités discrètes.

Une analyse didactique de cette question demande trois catégories d'études:

1. l'étude des manuels
2. l'étude des erreurs des élèves
3. l'étude des conceptions des enseignants sur les erreurs des élèves dans les calculs écrits

Notre exposé sera consacré principalement à l'étude des erreurs d'élèves de cinquième et sixième année de la scolarité élémentaire . Les erreurs dont il sera question ont été recueillies au moyen d'épreuves collectives papier-crayon y compris les brouillons que les élèves ont utilisés pour produire leurs résultats. Ce sont ces brouillons qui sont pris en considération à différents niveaux d'analyses des procédures des élèves.

Ces niveaux d'analyse sont respectivement la description, la classification , l'interprétation et l'explication des erreurs observées.

Pour comprendre les erreurs des élèves il est indispensable de faire intervenir d'abord l'examen des manuels scolaires . Les manuels actuellement en vigueur dans les écoles de Suisse romande présentent , en résumé, les caractéristiques suivantes:

- le sujet traité n'est pas seulement le calcul écrit de l'opération de division, mais la division elle-même; plus exactement la construction de l'algorithme est une occasion de traiter des notions générales comme :
- les évaluations par approximation des résultats du calcul
- la proportionnalité

- les nombres décimaux

La progression de l'apprentissage conçue par les manuels (depuis la quatrième année jusqu'à la sixième année incluse) passe par les étapes suivantes de l'enseignement :

- introduction du "sens" (signification) par une situation de partage : distribution d'unités discrètes.
- introduction des relations multiplicatives par des situations d'échanges, afin de donner du sens à la "sous-opération" de multiplication.
- étude de la proportionnalité des termes de la division .
- utilisation des propriétés de la division pour le calcul mental.

Les manuels ont été récemment révisés; il est significatif de noter qu'avec l'expérience dans les classes, l'objectif initial qui consistait à faire évoluer chaque élève d'une forme de calcul (soustractions successives) à une autre plus élaborée (dite forme économique parce que plus concise) est présenté de façon moins définitive : une plus grande souplesse est recommandée. C'est reconnaitre qu'une progression didactique ne correspond pas à une hiérarchie de complexité d'apprentissages. Comparons par exemple la forme de division la plus élaborée , dite "économique", à la forme plus élémentaire dite "par soustractions successives". Dans la forme élaborée, il est plus simple de former le quotient par la détermination successive des chiffres qui composent son écriture que de le former par le comptage, une à une , du nombre de fois que le diviseur a pu être soustrait du dividende. Par contre , il est plus complexe de décomposer le dividende en une suite de " dividendes partiels " pris successivement en considération selon des règles précises que de considérer le dividende puis la suite des restes que l'on obtient au fur et à mesure des soustractions. Il n'y a pas une hiérarchie de complexité claire entre ces différentes formes présentées dans les manuels.

L'examen des travaux des élèves montre surtout l'apparition de formes nouvelles, non décrites dans les manuels mais produites par les élèves. Les plus remarquées sont des formes de division erronées. Ce sont en général des formes hybrides, comme le montreront les exemples; hybrides au sens où elles sont entre les deux formes présentées ("par soustractions successives" et " économiques") en retenant et combinant des éléments et de l'une et de l'autre.

On peut considérer ces formes hybrides comme des formes transitoires, témoignant d'élèves en train d'apprendre et de "faire leurs propres expériences" en matière de calculs de division. La question est alors de se demander comment on peut les faire évoluer vers des formes plus cohérentes. Choisir un ordre de succession entre

les deux formes qui se sont malencontreusement mélangées ce serait insister sur la stabilité de ces formes de calcul plutôt que sur leur évolution. La question est de savoir comment l'élève pourra se lancer dans des calculs plus sophistiqués.
Que pouvons-nous affirmer sur la façon dont les élèves évoluent dans leurs calculs ? Peu de choses encore, et dans ce peu de choses , des indications contradictoires nous montrent que pour une part ils évoluent peu, lentement, et résistent longtemps devant l'adoption de nouvelles procédures , que pour une autre part ils évoluent de façon relativement autonome et font preuve de faculté d'adaptation comme en témoignent les formes hybrides que nous venons de citer en exemple.
Il est intéressant de se poser deux questions à propos de ces formes : comment sont-elles venues à l'esprit des élèves? et comment ces élèves pourraient-ils se corriger de manière spontanée. Le qualificatif "spontané" est important ici car il lie les deux questions posées à l'instant : trouver une réponse à la seconde question fait que l'enseignant n'aura pas à se préoccuper d'interdire à ses élèves d'essayer leurs "trucs" de calculs; mais de manière spontanée n'exclue pas l'enseignant, cela va de soi.
Les manuels actuels proposent une réponse à la deuxième question : il s'agit pour le maître de demander à l'élève d'expliquer et de justifier son calcul, voire même la disposition qu'il a adoptée. Le raisonnement sous-jacent est que la simple demande d'explication de la part de l'enseignant devrait amener l'élève à changer sa façon de faire. En fait , il n'est pas toujours possible à un élève (ni à quiconque d'ailleurs) de justifier et d'expliquer comment il a trouvé une réponse, quand bien même cette réponse serait correcte, avec conscience de son bien-fondé. D'autre part cette exigence faite à l'élève de ne fournir de réponse que justifiable et explicable pourrait être plus paralysante que stimulante. Imaginons que nous ayons demandé à un élève produisant un de ces hybrides de s'expliquer et qu'il n'aie pas pu le faire. Il sera alors bien convaincu qu'il faut faire autrement, mais il lui restera à savoir que faire d'autre, comment faire autrement et comment s'assurer que ce changement va effectivement corriger son erreur. Notre expérience nous montre que les élèves préfèrent le plus souvent chercher à glaner des indications chez le maître plutot que se risquer à nouveau.

Une autre étape nécessaire à la compréhension des erreurs des élèves consiste à classer les erreurs observées sur les différentes épreuves.

Description des épreuves
Nous avons soumis à 6 classes de Cinquième et Sixième années une épreuve écrite en quatre parties.

Les deux premières concernaient le calcul écrit lui-même et ont permis d'entreprendre une première classification des erreurs.
Les résultats de la première épreuve de cacul ont permis de construire la typologie suivante :

1. Erreurs en rapport avec le contrôle des relations Dividende / Diviseur / Quotient / Reste. à chaque étape de l'algorithme.

Si l'on considère l'ordre dans lequel l'élève traite chaque étape, on peut répertorier les erreurs suivantes:

1.1 Traitement du dividende
a. Choix d'un Dividende plus grand ou égal à dix fois le diviseur
implique

Quotient plus grand que 9 ; d'où :
- inscription d'un nombre à deux chiffres au quotient
- inscription de 9 au quotient

b.Dividende plus petit que le diviseur:
implique 0 au Quotient

erreurs : - saut à la colonne suivante sans 0 au Quotient
- inscription de 1 au Quotient

1.2 Traitement du Quotient
a. Choix d'un Quotient inférieur
implique un Reste plus grand que le Diviseur; d'où - division du reste avec un deuxième chiffre au quotient pour un même Dividende
- reprise du cycle de la division avec
+ un nombre à deux chiffres au quotient
+ une suite de 9 au quotient

1.3 Traitement du Diviseur

- le diviseur est découpé en une suite de chiffres non-emboîtés

1.4 Traitement du Reste

a. Reste final plus grand ou égal au Diviseur
b. Reste final divisé bien que plus petit que diviseur, d'où un 0 supplémentaire au Quotient

2. Erreurs en rapport avec les opérations intermédiaires

2.1 Erreurs de soustraction

- impliquent des dividendes erronés , d'où voir 1.1
- impliquent un reste inexact

2.2 Erreurs de multiplication

a. recherche du produit par additions successives :
- erreur sur le nombre d'additions.
- erreur d'addition

b. livret

3. Erreurs en rapport avec le placement des chiffres dans le diagramme de la division.

3.1 Placement du dividende
- n'abaisse pas le chiffre de la colonne ; d'où
- arrêt
- 0 au quotient
- abaisse un 0

3.2 Inversions

- le reste est écrit à l'emplacement du quotient
- le produit est écrit à l'emplacement du quotient
- le dividende est découpé en allant de droite à gauche

La troisième partie des épreuves écrites consistait à compléter trois divisions :
9072 : ... = ...
... : 38 = ...
... : ... = 95
Nous avons constaté de fréquentes erreurs à ces items où l'élève reconstitue un dividende par une sorte de multiplication chiffre à chiffre qui a un rapport évident avec la procédure de division.
Ce qui importe comme observation c'est qu'ils essayent de résoudre le problème dans le cadre de la division, comme l'indique la forme de la donnée.

La quatrième partie consistait en quatre énoncés :
a. Dans 552 il va 3 fois 24, c'est sûr . Mais on peut y mettre encore beaucoup de fois 24. En tout combien de fois peut-on mettre 24 sans 552 ?
b. De 728 j'ai enlevé 8 fois 26 et il me reste 520. Combien de fois faut-il additionner 26 pour avoir 728 ?
c. De 800 j'ai enlevé 2 fois 320 et il me reste 5 fois 32. Puis j'ai divisé 800 par 32 et j'ai trouvé 15. Ai-je fait juste ?

d. Dans 86 il va 3 fois 27. Et dans 864 ?
Le résultat de cette passation est que les élèves ne recourent vraiment à un calcul de division que dans l'exercice c. La mention de la division est présente dans l'énoncé.
En général les élèves refont tous les calculs mentionnés :
800 - 640 = 160
5 . 32 = 160
800 : 32 = 25
Très peu songent à faire la vérification de la division 800 : 32 = 15 comme ils l'ont appris, c'est-à-dire par le produit 32 . 15 = ...

En ce qui concerne les autres énoncés ils sont majoritairement traités comme des problèmes de multiplication et donnent lieu à des calculs répétés (par tâtonement) de multiples, comme si le mot "fois" était l'indice de leur résolution.
Ces élèves viennent d'apprendre , ou de répéter, la division; ils n'ont pas de peine à calculer, mais ils ne l'emploient pas dans des énoncés qui reprennent pourtant les termes consacrés du calcul. Ils font même au contraire appel à un autre calcul, celui de la multiplication, qu'ils réitèrent beaucoup de fois. Mais quand il faudrait penser à employer la multiplication (... : .... = 95) ils ne semblent pas l'envisager ."
Ces premiers résultats nous ont amenés à construire un second jeu d'épreuves auprès d'élèves de même niveau mais d'un autre établissement scolaire. Les tâches proposées visaient à mieux comprendre les relations que les élèves font entre multiplication et division. Nous avons alors constaté combien ces rapports sont centraux dans les tâches que nous avons proposées et combien ils permettent de lier les résultats des épreuves les uns aux autres. En particulier les informations obtenues à la seconde épreuve nous ont aidés dans l'interprétation des erreurs observées, lors de la première épreuve, dans les tâches d'effectuation de calcul mêmes. Nous espérons éclairer ce point particulièrement important lors de notre présentation.

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BRIAN'S NUMBER LINE REPRESENTATION OF FRACTIONS

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This report looks at cases of successful and unsuccessful student mathematical performance, and tests the explanatory power of the Lakoff/Johnson notion of basic metaphors which may lie at the root of even our most abstract and sophisticated ideas. In particular, it describes and analyzes, from the perspective of metaphor, a method used by Brian, a sixth grader, in his attempt to place fractions on a number line.

Lakoff, Johnson, and others have developed the idea that even our most sophisticated and abstract concepts have their foundation in quite concrete experiential metaphors learned in early childhood. One of the earliest uses of this idea was made by Stephen Young (1982). The 1980 Preliminary Scholastic Aptitude Test contained a mathematical problem involving pyramids, on which the answer keyed as "correct" was in fact incorrect, because all of the experts had solved the problem incorrectly. One student in Florida, taking the test, immediately saw the correct answer, and was unshakably confident about his choice. Although the student made no use of concrete models, and did not carry out any calculations, he was sure he was right. Young set himself the task of finding some basic metaphor, and corresponding mental representation, that would lead to such certainty, and was indeed able to find one (Young, 1982; Davis, 1984; see also Davis, 1986). We obviously do not know exactly what was in the mind of the Florida student, but Young proved a kind of "existence theorem": there does exist at least one mental representation, based upon one concrete experiential metaphor, that would lead any of us to equal certainty, without calculations and without the use of concrete models.

Other examples abound. Cobb, Wood, Yackel, and McNeal (in press) report the case of two second-grade children, Anne and Jack, trying to add up 12+12+12+12+12. Anne, following a rote algorithm (with no apparent concern for the meaning of the symbols or the procedures), correctly adds 2+2+2+2+2, gets 10, writes what she calls the "oh" in the proper place, "takes away" the 1, and writes it as a "carry". Jack questions her, objecting that it's not a "one" and an "oh", it's a *ten*. The obvious presumption is that Jack knows the idea of "taking one away from ten" in concrete terms (as he almost certainly must), based upon the experience of having ten objects (pebbles, say, or buttons), and getting the answer *nine*. Neither Anne nor the teacher ever recognizes Jack's apparent basic metaphor, and neither connects with Jack's thinking. Probably Jack has also had experiences that would make available to him an alternative metaphor - having ten pennies, and trading them for one dime - but he shows no evidence of retrieving this metaphor, and neither Anne nor the teacher allude to it. The communication is unsuccessful.

In our own videotaping of students as they work on mathematics problems, we have numerous instances where an inappropriately-chosen basic metaphor (or one employed incorrectly) gives some student special intellectual difficulties.In Maher and Alston (in press),we see two second grade students, Dana (D) and Stephanie (S) working on the problem:

> **Stephen has a white shirt, a blue shirt and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?**

The dialogue between them apparently made them more aware of the question of "what goes with what" to make an outfit.

D: Yeah, well just put white with blue [*Dana draws* 5 *connecting lines between the rows of shirts and pants.*]
S: Ssshhh...Ok, yellow shirt...number three can be a yellow shirt. [*She draws* 3 *and* Y *over* W.]
D: It can't...yellow can't go with the white.

In fact, this difference (tasteful combinations you may actually be willing to wear versus all possible combinations)

may have led to a heightened awareness of the possibilities of seeing combinations instead of seeing shirts and seeing pants. Even though Stephanie says explicitly "It doesn't matter if it doesn't match as long as it can make outfits. It doesn't have to go with each other, Dana", she is unable to penetrate completely Stephanie's idea that only aesthetically-satisfactory matchings should be counted.

For the past 5 years we have been videotaping a boy, Brian, as he works on mathematics (Davis & Maher, 1990; Maher & Davis, 1990; Maher, Davis & Alston, in press). Our particular concern in this report is to describe a method used by Brian, when he was in the sixth grade, to place fractions on a number line. He offers a systematic procedure that is consistently retrieved and applied as he orders the fractions. Unfortunately, in the task-based interview in question, he was not asked to justify the method, so we do not know with certainty what mental representation was guiding his decisions, but there is more than enough evidence to match against a Young type of existence proof.

Pre-Interview: Representation of Fractions. Brian, as a part of his pre-interview conducted before he began a six-grade unit on fractions, was asked to place fractions on a number line. This seemed to be an unfamiliar idea to him.

I: Have you ever used a number line before?
B: Number line?
I: Yeah - Do you know what a number line is?
B: No - I don't think so.

The interviewer drew a line segment, placed 0 and 2 on it, and asked Brian where he would put 1. Brian responded correctly and then, as the interviewer asked him, also added 3, 4, 5, and 6 to the drawing of the line, remarking that they kept going. The two agreed that this would "go on forever" and then Brian was asked where he would place 1/2 and 1/4 on the line.

B: If you wanted 1/2, you take the zero to the one - and 1/2 of it - right in a row and that would be half of - in here - so it's equal parts. That's what it is.
I: What about 1/4. Where would it go?
B: (Pauses, then points his marker toward the 4.) Um, maybe - I'm not sure, but maybe - No. It wouldn't go there (Moves marker up and down the line.) [In tentatively pointing to the four, Brian gives us the first clue as to what basic metaphor he has in mind. We return to this later, after collecting more evidence.]
I: What are you thinking? Where do you think it should go?
B: I thought it would go right here (Around the 4.) But - somewhere about in here, I think (Marks a point, when prompted, between the zero and his point for 1/2.)

The interviewer asked Brian why he had chosen that point, and he answered:

B: Like from zero - and maybe you keep going - 1/4 of it and like 1/3 - then 1/2.

When asked to place 1/3 on the line, Brian placed a mark between the points for 1/4 and 1/2.

Thus far -- except for Brian's consideration of 4 -- one might imagine that Brian is using the *size* of the rational number, quite correctly, as his basic idea. But wait!

The interviewer then asked where he would put 2/3, and Brian placed a mark about half way between the 1 and the 2 on the line, but then expressed uncertainty about what he had done.

I: What are you thinking?
B: (Pointing back to the unit fractions.) Like if I was going over this, I could say 1/4, like 1/3 - but here (Pointing back to the mark between 1 and 2) if I wanted to write 2/4 and then 2/3 - like two and then - one.
I: What is 2/4? What does 2/4 mean?
B: When you have two things - or four things and two parts out of it.

The interviewer reminded Brian of a problem he had done earlier in the interview, in which he had built fourths of a

unit from pattern blocks. Brian agreed and pointed to the concrete representation that he had constructed in which the 2/4 was represented as two out of four red trapezoids which covered the two yellow hexagons that he had chosen to represent his unit.

I: So that's 2/4 there [i.e., with the Pattern Blocks]. Where would it go on the number line?
B: Maybe here (Places a mark between the 1 and the mark he made for 2/3)

By now we have enough evidence to begin a tentative identification of the basic metaphor on which Brian is building. It is based not upon *size*, but rather upon *classification*. What concrete experience could Brian be drawing upon? One possibility is the labeling of street addresses. If an address says 603 Second Avenue, Apt. 215, you consider first the "Second Avenue" to identify the street, then the "603" to find the building, then the "Apt. 215" to find the correct apartment. In his uncertainty in hovering near the 4 on the number line, Brian may be showing us not merely that he is using classification, but also that he is unsure as to whether the first choice should be based upon the denominator (hence the 4), or the numerator (leading to a choice of 1 instead of 4).

I: Why? Can you explain why you put it there?
B: Here you have one whole. That would be this (Points to the mark for 1 on the line.)...Like the next part of it. (Points to the mark for 2/4 just beyond the mark for 1.)

When asked where 4/3 would be on the line, Brian immediately placed a mark between the 3 and the 4 on the line. Brian explained his decision about placing the fractions on the line on three different occasions during the interview. Each explanation was consistent and indicated that for him, this was a sorting task with the numbers positioned along the line according to two criteria. The first focused on the numerator. According to its value, he sorted the fractions into sets which were to be placed somewhere in the unit to the left of that point on the line.

I: Can you say why you decided to put it (the 4/3) there?
B: Like one (He pointed to the unit between zero and one on the line.) Like 1/2. Then you went to the two's and you started in here. (He points to the fractions with 2 as numerator which he had placed between 1 and 2 on the line.) Maybe you got 3. (He pointed to the space between 2 and 3 where no fractions had been marked.) And here 4. (He pointed to 4/3 which he had placed between 3 and 4.)

The second criterion was the denominator. Brian seemed certain that the larger that number, given the same numerator, the closer the point should be placed to the unit position that was one less than the numerator. At this point he apparently did make use of size. Earlier in the interview, when asked which was larger 1/3 or 1/4, Brian had defended his answer (that 1/3 was larger).

B: Because if you take 1/3 (He holds up a blue parallelogram as one part of the yellow hexagon that represented his unit.) This would be one part of it, and if you wanted to fill this (the hexagon) in with four - it would be smaller.

This representation seemed to influence his decision about placing each set of fractions having a common numerator on the line.

I: And the fractions that you put in? How did you decide where to put them in?
B: (Pointing to the unit fractions.) Like here would be 1/2, and like if you keep going down it would be 1/3 and then 1/4 and then you keep getting closer to 1/6.

Further indication that the system that Brian had built in order to make sense of the mapping of fractions to certain points on a number line was inconsistent with his ideas about fractions as parts of a unit came when he was asked where two halves would go on the line:

B: Two halves?
I: Would you write two halves up here (as a number) so we can see the problem?
B: Two (over) one?
I: Two halves. Two - two.

Brian seemed to be uncertain about the symbolic representation for two halves but after this exchange wrote the number 2/2 on the paper in response to the interviewer.

I: Two halves. What is two halves?
B: Two wholes?
I: No. What is two halves?
B: Two halves. Like take two things and cut them in half.
I: If I gave you two halves, how much would I give you altogether?
B: One whole.
I: So where are you going to put two halves?
B: Maybe here. (Brian marks a point between 1 and 2 beyond the point that he had marked 2/3.)
I: So you think it should go between the 1 and the 2?
B: Yeah.

Brian clearly has a conflict: Is he supposed to be dealing with *numbers* or with *numerals?* In terms of numbers, 2/2 is the same as 1, but the numerals are quite obviously not at all the same.

Conclusions

Following Lakoff, we could argue that abstract ideas can often be traced to some specific metaphor, based upon previous experience. A student, faced with some mathematical problem, is more likely to deal with it successfully if he or she builds upon a correctly-selected basic metaphor, and is likely to be led astray if an inappropriate basic metaphor is selected. If a teacher (or fellow student) can recognize these basic metaphors, he or she can communicate more clearly with students, especially in cases where a student may be retrieving, and attempting to build upon, an inappropriately-chosen basic metaphor.

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PUPILS' NEEDS FOR CONVICTION AND EXPLANATION WITHIN THE CONTEXT OF GEOMETRY

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Summary

This article reports empirical research which indicates that although the majority of pupils' needs for personal conviction is frequently satisfied by quasi-empirical means, they nevertheless exhibit an independent need for explanation which can only be satisfied by some sort of logico- deductive argument. Pupils' needs in this respect and their fulfillment are also compared with those of mathematicians, and a case is then made out that the explanatory function of proof is not only potentially more meaningful to pupils than the verification function, but also more intellectually honest.

1. Introduction

The problems that pupils have with perceiving a **need** for proof is well-known to all high school teachers and is identified without exception in all educational research as a major problem in the teaching of proof. Who has not yet experienced frustration when confronted by pupils asking "why *do we have to prove this?*" According to Afanasjewa in Freudenthal (1958:29) pupils' problems with proof should not simply be attributed to slow cognitive development (e.g. an inability to reason logically), but also that they may not see the **function** (meaning, purpose and usefulness) of proof. In fact, several studies have shown that very young children are quite capable of logical reasoning in situations which are real and **meaningful** to them (Hewson, 1977; Donaldson, 1979). Furthermore, attempts by researchers to teach logic to pupils have frequently provided no statistically significant differences in pupils' performance and appreciation of proof (e.g. Deer, 1969; Mueller, 1975). More than anything else, it seems that the fundamental issue at hand is the appropriate **negotiation** of the various functions of proof to pupils.

This raises two questions which were investigated by the author in De Villiers (1990a):

(1) what **functions** does proof have within mathematics itself which can potentially be utilized in the mathematics classroom to make proof a more meaningful activity?

(2) what cognitive **needs** do pupils spontaneously exhibit, or can be created, so that the functionality of proof can be illustrated in the fulfillment of those needs?

Regarding the first question, Bell (1976:24) assigned the following three functions to a proof, namely:

(a) verification (justification or conviction); concerned with the truth of a proposition

(b) illumination (explanation); conveying insight into why a proposition is true

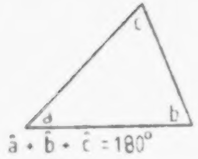
(c) systematization; the organization of results into a deductive system of axioms, major concepts and theorems

Besides the above functions, proof for a mathematician often also fulfills a function of discovery and/or communication as discussed in De Villiers (1990a & b). This paper, however, will only deal with the author's investigation of pupils' cognitive needs with respect to conviction and explanation within the context of geometry.

2. Pupils' cognitive need for conviction

The term conviction is used here as a synonym for justification, verification and certainty. In 1986 519 Std.7 to Std.10 pupils (grades 9 to 12) from a Technical High School were given 42 geometric statements from the formally prescribed geometry syllabus (1983) in South Africa for Std.7 to Std.10, and asked to make the following judgements with respect to each of them: Code 1: Believe it is true from own conviction; Code 2: Believe it is true because it appears in the textbook or because the teacher said so; Code 3: Do not know whether it is true or not; Code 4: Do not think it is true; Code 0 : Unanswered.

Code	0	1	2	3	4
1. The sum of the interior angles of a triangle is 180°					
Std.7	0	37.1	50.5	6.7	5.7
Std.8	0	23.1	71.4	3.4	2.1
Std.9	0.7	22.8	71.0	3.4	2.1
Std.10	0.8	18.7	74.8	3.3	2.4
2. The exterior angle of a triangle is equal to the sum of the opposite interior angles					
Std.7	2.9	21.8	64.8	4.8	5.7
Std.8	0.7	14.2	76.9	4.1	4.1
Std.9	2.8	11.7	78.6	4.8	2.1
Std.10	1.7	14.6	69.9	8.9	4.9



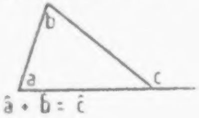


TABLE 1

Table 1 gives an example of typical percentages of responses to the statements (note that the given two were informally treated in Std.6 and then formally proved in Std.7). As Code 2 responses were in all cases far greater than Code 1 responses, the overriding (and extremely worrisome) impression was created that: *the certainty or conviction of the majority of the pupils with respect to prescribed statements seemed to be based on* **authoritarian** *grounds rather than on personal conviction.*

Also during 1986, 1465 Std.7 and Std.8 pupils from 10 different schools in a study by Smith (1987) and 494 Std.9 and Std.10 pupils from 4 different schools in a study by De Villiers & Njisane (1987) were firstly asked whether they were certain of the truth of the following two statements (which are proved in Std.7), and secondly on what grounds they based their certainty (or uncertainty):

(1) In an isosceles triangle the angles opposite the equal sides are equal.
(2) If two parallel lines are cut by a transversal, then the alternate angles are equal.

Although 88% of the pupils were certain of the truth of both these statements, only 7% indicated that they were certain because they could be proved, while the majority just repeated the statements as reasons or simply left the second parts unanswered. This seems to suggest that only for a minority of pupils, proof has the function of conviction/justification.

In order to gain additional information about pupils' needs for certainty and how they themselves would choose to obtain personal conviction, the author conducted a teaching experiment and a series of interviews with Std.6 to Std.10 pupils during 1987 and 1988. For this purpose pupils were placed in *new* or relatively *unknown* problem situations. The following finding was then made: the *majority of pupils spontaneously satisfied their need for personal conviction in such situations by means of quasi-empirical testing.* This finding was based on the following results:

- 94% of the std.7 pupils (total 32) in a teaching experiment *spontaneously* indicated that further quasi-empirical testing (i.e. construction and measurement) would satisfy their need for certainty with respect to the following geometric conjecture: *"If the midpoints of the adjacent sides of a quadrilateral ABCD are consecutively connected, then a parallelogram ABCD is formed (see Figure 1)"*

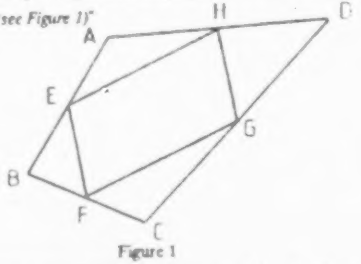


Figure 1

Peter recently connected the midpoints of a quadrilateral ABCD as shown above, and noticed that EFGH seemed to be a parallelogram. How would you make certain whether this was always true?

Typical responses were (freely translated from Afrikaans):

"Draw a million and test"

"Issue a law that the whole population should draw at least two quadrilaterals to test the result"

"Let a class of school children each draw a variety of quadrilaterals and test the result. If there are NO exceptions, it will always be possible."

- 8 out of 11 Std.6 to Std.10 pupils interviewed, also spontaneously obtained certainty with respect to the above conjecture by means of construction and measurement of a number of different quadrilaterals. (5 of these pupils were from Std.8 to Std.10). Only 3 pupils therefore explicitly chose deduction as a method of verification.
- In another interview, 3 out of 7 Std.9 and Std.10 pupils obtained certainty exclusively by means of construction and measurement with respect to their own (visually formulated) conjectures that the adjacent angles and diagonals of an isosceles trapezium were equal (see Figure 2). For example consider the following extract from the interview with Lara (Std.10):

I: "Is there anything in this figure that looks equal?"
L: "∠ A and ∠ D look the same, as well as ∠ B and ∠ C"
I: "Is there anything else which looks the same? ... (no answer) What about the diagonals?"
L: "The diagonals will be equal"
I: "Are you 100% certain that the angles and diagonals are equal?"
L: "I think I'm certain about the angles, but I'm not so sure about the diagonals ... perhaps reasonably certain about them"
I: "How will you make dead sure whether they are really equal?"
L: "By drawing them and measuring."

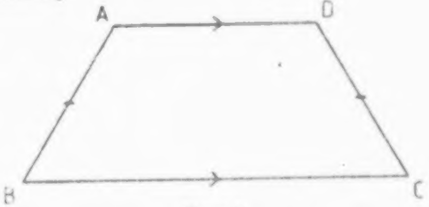


Figure 2

What conjectures about the above figure can you form? How would you make certain whether they were true or not?

These 3 pupils also did not display an additional need for the deductive verification of these results, as illustrated by the following extract from the interview with Lynette (Std.10) who had convinced herself by construction and measurement (freely translated from Afrikaans):

I: "If I would now give you a proof ... would that make you more certain that it is always true ... or are you at the moment sufficiently satisfied that it is always true?"
L: (with emphasis as indicated) "I am presently satisfied, since I observed it **myself**, and measured it **myself**. I am feeling satisfied because I did it **myself**."
I: "You have no need that I give you a proof to convince you further?"
L: "No, I convinced myself." (sounding very firm)

Two of the other four pupils were also asked to evaluate certain alternative definitions for the isosceles trapezia. In both cases they chose quasi-empirical testing rather than deductive proof. For instance, Martin (Std.9) reacted as follows when asked how he would make sure whether any cyclic quadrilateral with equal diagonals was an isosceles trapezium or not (freely translated from Afrikaans):

M: "One could draw a circle with two chords of equal length and intersecting, and then connecting the end points and measuring the base angles, and if the base angles are not equal, then it is not an isosceles trapezium."

The other student was also of the opinion that a mathematician would not, like himself, make a construction to test a definition (or conjecture), but only use logical deduction ("he would sit here with an *a* and *x* and prove it").

- In another interview, all 5 the Std. 7 to Std. 9 pupils chose construction and measurement to evaluate and obtain certainty about the validity of the following two conjectures "inscribed angles in a semi-circle are 90°" and "angles inscribed on the same chord are equal."

In the above examples, pupils did not always use accurate construction and measurement, but sometimes only rough drawings which they then simply evaluated visually (e.g. the first example). The above results were also not isolated observations, but has regularly been observed by the author in various other situations with different geometric conjectures. Of course, the finding based on these results is not new, and is confirmed by empirical research on the Van Hiele theory (e.g. Usiskin, 1982; Fuys, Geddes & Tischler, 1988). Schoenfeld (1986:243) also writes as follows: "*... most students from high school sophomore through college senior, all of whom has a full year of high school geometry, are naive empiricists ...*"

3. Pupils' cognitive need for explanation and its fulfillment

The following finding was made in this regard by the author (De Villiers, 1990a): *all the pupils who had convinced themselves by quasi-empirical testing still exhibited a need for* **explanation**, *which could only be satisfied by some sort of informal or formal logico-deductive argument.*

- Despite their certainty with respect to the conjecture that a parallelogram will always be formed by connecting the midpoints of the sides of any quadrilateral, all the Std.7 pupils in the teaching experiment exhibited an independent need for explanation, which was illustrated by responding extremely positive to the question: "You have now all convinced yourself that this conjecture is true, but would you like to know why it is true?"

The class then seemed to find the given deductive explanation quite satisfactory.

- All 8 of the Std.6 to Std.10 pupils who had chosen to use quasi-empirical testing to obtain conviction with respect to the previous conjecture showed an additional need for explanation. For example, consider the following extract from the interview with Vicky (Std.8) (freely translated from Afrikaans):

I: "Are you now dead sure that it will always be true?"
V: "Yes, now I'm certain."
I: "Will a logical proof make you more certain?"
V: "Yes, but ... not really. I'm quite certain."
I: "Do you rather have a need to know why it is true?"
V: "Yes"
I: "Do you really want to know why?"
V: "Yes, quite ... I'm inquisitive about it"

The following is another extract of an interview with two Std.6 pupils, Nadia & Lizelle:

I: "Good, you say you are now certain. You were convinced by a number of examples. But now I would like to ask you: Do you know why it is true?" ... Are you interested in knowing why it is true?"
L: "I would like to know"
I: "An explanation? A clarification?"
N & L: "Yes"

The interviewer then gave a logical explanation in terms of the result that the line connecting the midpoints of two sides of a triangle, is parallel to the third side.

I: "So does this explanation satisfy your need for explanation?"
N & L: "Yes" (sounding very satisfied)

- Despite displaying no further need for deductive verification, the 3 pupils who had used construction and measurement with respect to the given isosceles trapezium, still exhibited a need for explanation which had not been satisfied by their quasi-empirical approach. For example, consider the following extract from the interview with Lynette (Std.10) (freely translated from Afrikaans):

I: "Do you perhaps have a need to know why it is true?"
L: "Yes ... why is it true?"
I: "What I understand under the word 'why' is: can I explain it in terms of something else. Do you give the same meaning to it?"
L: "Yes"
I: "For which properties would you most like to have an explanation why it is true?"
L: "Why the diagonals are equal"

The interviewer than gave a deductive proof of the result in terms of the congruency of triangles ABC and DCB.

I: "Did this this explanation satisfy your need?"
L: "Yes ... to a degree" (sounding reasonably satisfied)

- Similarly the 5 pupils in the interview about the aforementioned circle theorems, still exhibited a need for explanation which had not been satisfied by their quasi-empirical approach; a need which was then only satisfied by the production of a logico-deductive argument. For example, consider the following extract from the interview with Lennart (Std.9) (freely translated from Afrikaans):

I: "How would you make certain ... Or are you already dead sure that those angles are also equal?"
L: "No"
I: "Now how would you yourself make sure if you went home this afternoon? Suppose you had an urgent need to make sure: how would you make sure? What would you do? ... Now I want your honest opinion. I do not want what you may think I am expecting of you - I need your personal opinion."
L: "I would perhaps make a larger drawing and work more accurately"
I: "You would not perhaps try to produce a proof for it yourself?"
L: "Yes, if I make larger drawings, and the angles continue to come out equal, then you could prove that if you have a chord connected to two points on the circle, then those angles will be equal"
I: "Do you feel that if you have drawn enough accurately and measured them, and suppose you would see that they were always equal, would that be sufficient proof that it is always true? Remember I am asking you your opinion."
L: "Yes, if I have done it quite a number of times"
I: "Do you have a need to see a proof for it?"
L: "No"
I: "But do you perhaps wonder why it is true? Do you perhaps have a need to know why those two angles are equal, rather than just knowing that they are equal?"
L: "Yes, I would very much like to know why it is true"
I: "Really?"
L: "Yes"

The interviewer then gave a logical proof of the result in terms of the "*exterior angle of a triangle*" theorem.

I: "Now does this argument explain to you why it is true? Does it satisfy your need for explanation? ... Does it satisfy you completely?"
L: "Yes, it satisfies me ... I think it is sufficient" (sounding quite satisfied)

Although Lennart (Std.9) clearly had no need for a logical proof within the context of *verification*, he nevertheless displayed an independent need for *explanation*, which was then satisfied by a logical proof.

It should , however, also be pointed out that some of these pupils did indicate that the given logico-deductive argument had also further increased their confidence/certainty, although it had already been very high due to their quasi-empirical approach. Although it was not an original research question of the study reported here, the author has also on occasion with other children in other situations found that informal arguments often offer sufficient explanation. For example, the folding of an isosceles triangle along its line of symmetry to explain why the base angles are equal, or the rotation of a line around a fixed point causing it to swing out equal angles on both sides to explain why opposite angles in two intersecting lines are equal.

3o the pupils' aforementioned needs and their fulfillment compare to those of professional
aticians?

ematicians' cognitive need for conviction and its fulfillment

ı some people **may just** dismiss the aforementioned pupils' lack of need for deductive proof as a means of on, by **simply claiming that** they do not understand the real nature of mathematical proof and the insufficiency empirical evidence. **In fact**, probably most teachers of mathematics believe that a proof for the mathemati- vides **absolute certainty and** that it is therefore the **absolute** authority in the establishment of the validity of a re. They **seem to hold the naive** view described by Davis & Hersh (1986:65) that behind each theorem in the ıtical literature there **stands** a sequence of logical transformations moving from hypothesis to conclusion, ab- omprehensible, and irrefutably guaranteeing truth. However, this view is completely false. Proof is not necess- erequisite for **conviction** nor does it guarantee truth: to the contrary, **conviction is probably far more fre- ı prerequisite for the finding of a proof** (*For what other, weird and obscure reasons, would we then sometimes onths or years to prove certain conjectures, if we weren't already convinced of their truth?*).

ointed out by **Bell** (1976:24) this view of verification/conviction being the main function of proof *"avoids con- ı of the real nature of proof"*, since conviction in mathematics is often obtained *"by quite other means than that ing a logical proof."* Research mathematicians for instance seldom scrutinize the published proofs of results , but are rather led by the established **authority** of the author, the testing of special cases and an informal on whether *"the methods and result fit in, seem reasonable ..."* (Davis & Hersh, 1986:67).

-known George Polya (1954:83-84) also writes as follows that conviction often precedes a proof:

> *ving verified the theorem in several particular cases, we gathered strong inductive evidence for it. The induc- hase overcame our initial suspicion and gave us a strong* **confidence** *in the theorem. Without such* **confidence** *uld have scarcely found the courage to undertake the proof which did not look at all a routine job. When you satisfied yourself that the theorem is true, you start* **proving** *it."* (bold added)

ions like the above where conviction actually provides the **motivation** for a proof, the function of such a proof ıathematician **clearly** cannot be that of verification/conviction, but has to be looked for in terms of other func- proof, e.g. **explanation**.

lute certainty **also does not** exist in real mathematical research, and personal conviction usually depends on **nation of intuition, quasi-empirical** verification and the existence of a logical (but not necessarily rigourous) ı fact, a very **high level of conviction** may sometimes be reached even in the absence of a proof. For instance, discussion of the *"heuristic evidence"* in support of the still unproved twin prime pair theorem and the famous n Hypothesis, Davis & Hersh (1983:369) conclude that this evidence is *"so strong that it carries conviction even rigorous proof."*

thematicians' cognitive need for explanation and its fulfillment

:h it is possible to **achieve** quite a high level of confidence in the validity of a conjecture by means of quasi-em- erification (e.g. **accurate** constructions and measurement; numerical substitution, etc.), this generally provides factory **explanation why** *it may be true*. It merely confirms that it is true, and even though the consideration of ıd more examples **may** increase one's confidence even more, it gives no psychological satisfactory sense of il- ion, i.e. an **insight** or **understanding** into how it is the consequence of other familiar results. For instance, des- convincing **heuristic evidence** in support of the earlier mentioned Riemann Hypothesis, one may still have a ; need for **explanation** as stated by Davis & Hersh (1983:368):

> *ı interesting to ask, in a context such as this, why we still feel the need for a proof ... It seems clear that we want oof because ... if something is true and we can't deduce it in this way, this is a sign of a lack of understanding our part. We believe, in other words, that a proof would be a way of understanding* **why** *the Riemann conjecture ue, which is something more than just knowing from convincing heuristic reasoning that it is true."*

ly Gale (1990:4) also clearly emphasized as follows, with reference to Feigenbaum's experimental discoveries al geometry, that the function of their eventual proofs was that of explanation and not that of verification at all:

> *"Lanford and other mathematicians were not trying to* **validate** *Feigenbaum's results any more than, say, Newton was trying to validate the discoveries of Kepler on the planetary orbits. In both cases the validity of the results was never in question. What was missing was the* **explanation**. *Why were the orbits ellipses? Why did they satisfy these particular relations? ... there's a world of difference between validating and explaining."* (bold added))

Thus, in most cases when the results concerned are intuitively self-evident and/or they are supported by convincing quasi-empirical or heuristic evidence, the function of proof for mathematicians is certainly not that of verification, but rather that of explanation. It is not a question of *"making sure"*, but rather a question of *"explaining why"*. Furthermore, for probably most mathematicians the clarification/explanation aspect of a proof is generally of greater **importance** than the aspect of verification. For instance, the well-known Paul Halmos stated some time ago that although the computer-assisted proof of the four colour theorem by Appel & Haken convinced him that it was true, he would still personally prefer a proof which also gives an *"understanding"* (Albers, 1982:239-240). Also to Manin (1981:107) and Bell (1976:24), explanation is a criterion for a *"good"* proof when stating respectively that it is *"one which makes us wiser"* and that it is expected *"to convey an insight into* **why** *the proposition is true."*

From the preceding discussion, it should therefore be clear that there is not such a great difference between the aforementioned pupils' basic needs for conviction and explanation and their fulfillment, and those of professional mathematicians. To summarise: pupils are easily convinced by authority, but so are mathematicians. Pupils gain personal conviction by means of intuition and/or quasi-empirical testing, but so do mathematicians. Pupils empirically check already proven statements to further increase their confidence (e.g. Fischbein, 1982), but so do mathematicians who are only too aware of the inadequacies of deductive proof, not only from Gödel's theorem and the host of examples of false and incomplete proofs from the history of mathematics, but also from personal experience. Finally, like mathematicians, pupils also exhibit an independent need for explanation which only seems to be satisfied by the production of some sort of logico-deductive argument. Of course, there is a vast difference in the level of sophistication of a mathematician's approach and those of the aforementioned pupils, but there is nevertheless a remarkable comparison.

5. Some brief concluding comments about the teaching of proof within the context of geometry

Traditionally the role and function of proof in the classroom has either been completely ignored (the fact that it is in the syllabus and will be examined is considered sufficient reason for its treatment in class), or it has been presented as a means of obtaining certainty (i.e. within the context of verification/conviction). However, as pointed out in this article, mathematicians often construct proofs for quite other reasons than that of verification/conviction. The popular idea of many mathematics teachers that conviction is a *one-to-one mapping* of deductive proof (i.e. a **bijective function**) should therefore be completely abandoned; *conviction is not gained exclusively from proof alone nor is the only function of proof that of verification/conviction*. Not only does such an approach represent complete intellectual dishonesty, but as shown in this article it also does not make sense to many pupils, especially where self-evident or easily empirically verifiable statements are concerned.

Rather than one-sidedly focussing only on proof as a means of verification, the more fundamental function of proof as a means of explanation could be exploited to present proof as a meaningful activity to pupils. For instance, although pupils may have no need for further (deductive) verification, their need for explanation may be utilized as a meaningful context for the presentation of a proof. Rather than focussing on the inadequacy of quasi-empirical methods with regard to verification and (falsely) presenting proof as the only legitimate and absolute means of verification, the focus could fall on their inadequacy with respect to explanation. The author has also found it useful to use the term (deductive) *"explanation"* in such situations, and that the term (deductive) *"proof"* is initially restricted to those situations where pupils explicitly exhibit a need for verification/conviction (*"to make sure"*).

By stressing the explanatory function instead of the verification function in the case of self-evident statements and/or statements that have already in previous years been verified by quasi-empirical means and/or authoritatively announced as true, the author has found quite considerable improvement in pupils' meaningful appreciation for the role and function of proof. This was achieved simply by explicitly telling pupils that in such cases we were not really interested in verifying the truth of the statements concerned (we already knew they were true), but in trying rather to find out **why** they were true (i.e. upon which characteristic they depended or how they were the consequence of other results).

The author is furthermore of the opinion that, in cases where pupils express a need for explanation, strict preference should be given to logico-deductive arguments which really satisfy their need for explanation (compare Hanna, 1989), and must there be guarded against proofs which may not easily fulfill this function (e.g. analytical proofs). Informal arguments as well transformation proofs should also be considered in this regard, since it is suspected that in some cases (and at certain levels) an informal argument would be sufficient explanation, while some transformation proofs might be experienced as more explanatory than the corresponding Euclidean proofs. But this is a question of extensive further empirical research, since it was not adequately covered by the research reported in this article.

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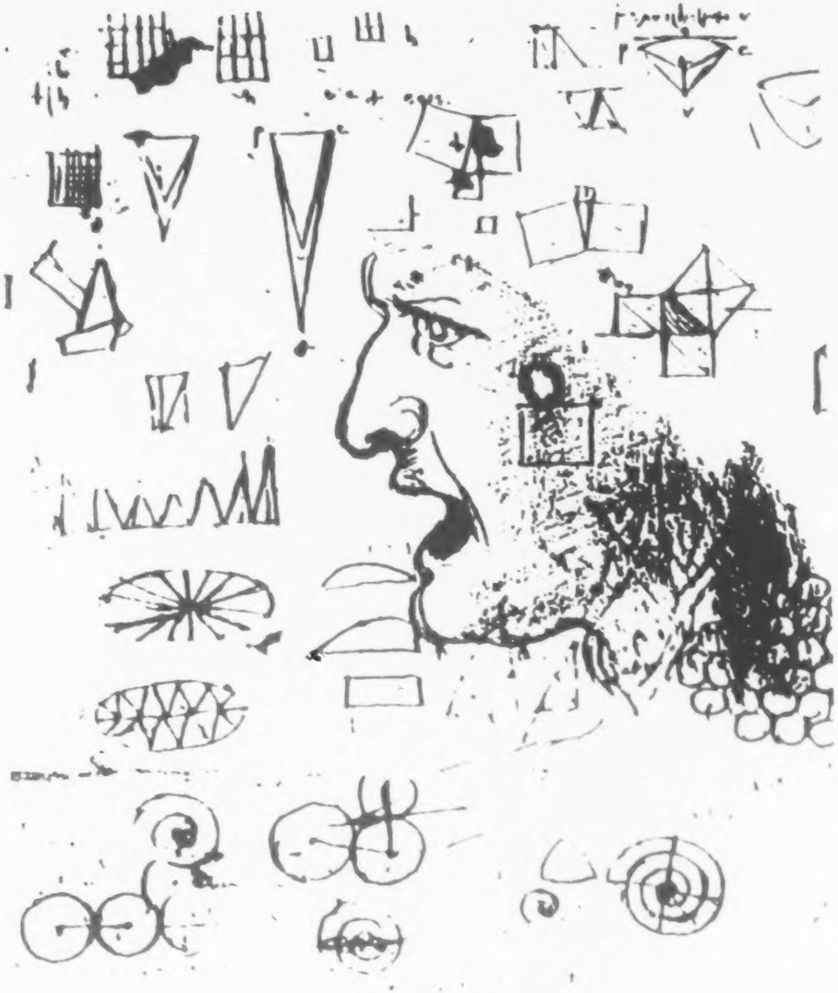

Leonardo da Vinci, Codice Atlantico
(in L. Villani, Leonardo a Milano, Musumeci Ed.)